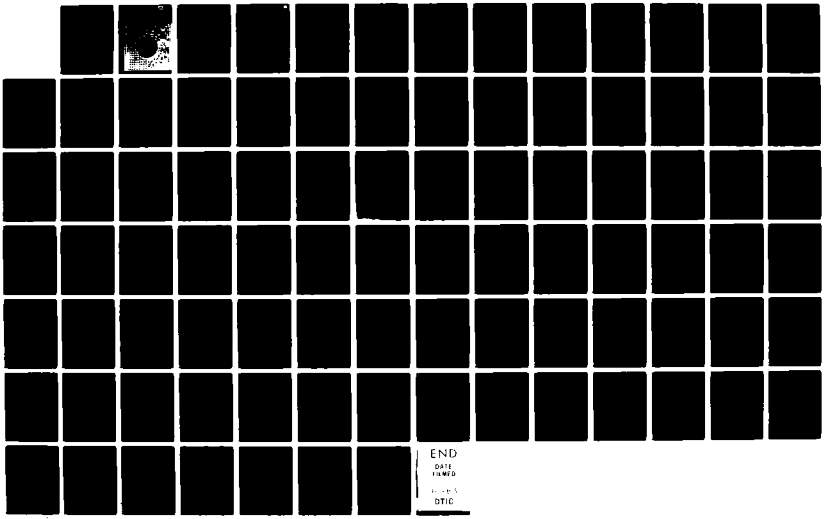

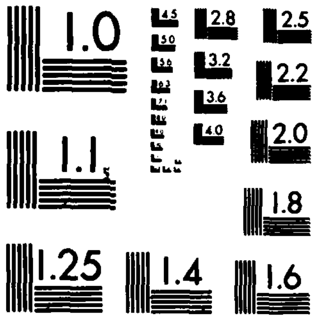

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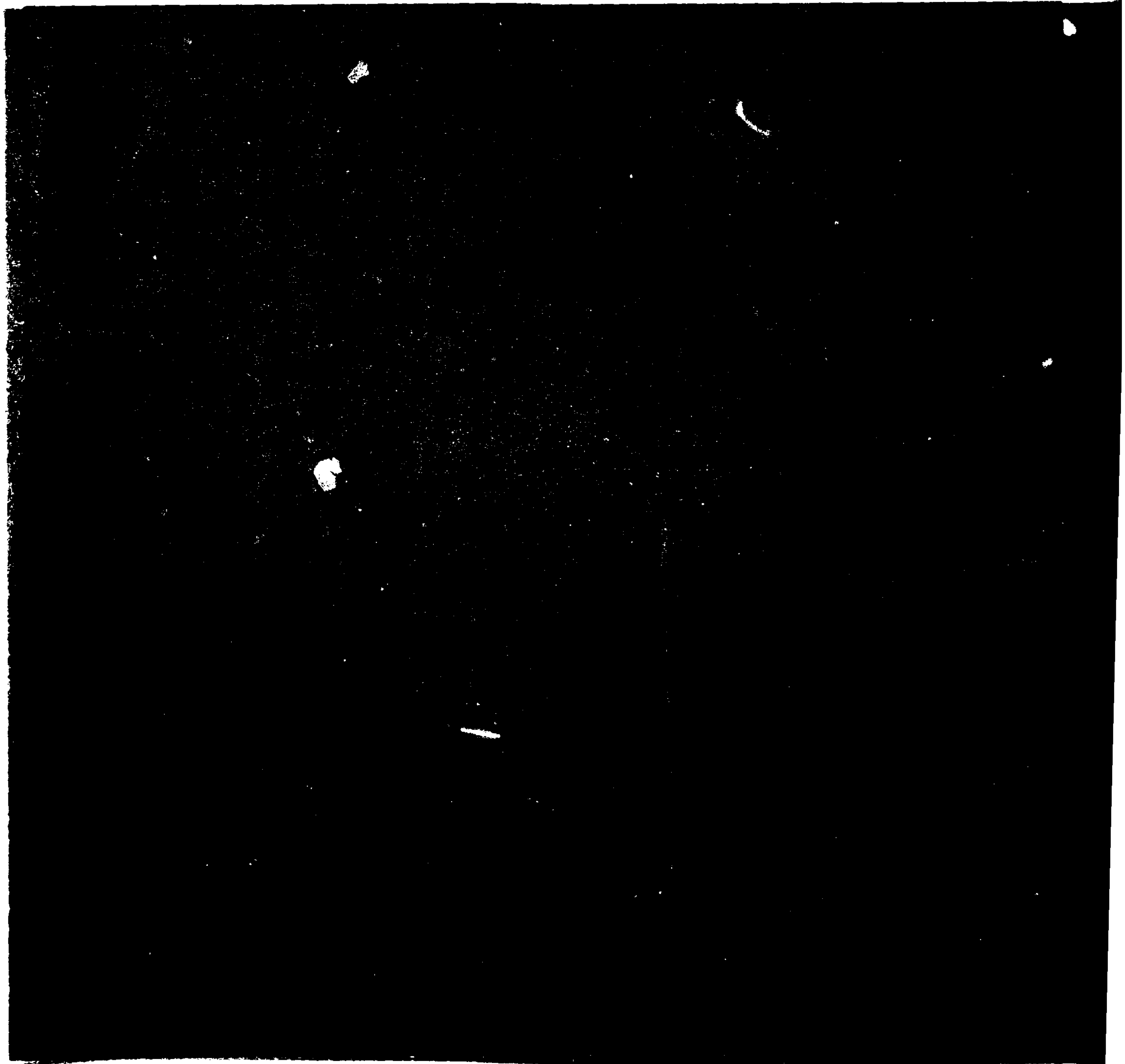

For conversion of SI metric units to U.S./British customary units of measurement consult ASTM Standard E380, Metric Practice Guide, *published by the American Society for Testing and Materials, 1916 Race St., Philadelphia, Pa. 19103.*

Cover: *A circular boundary modeled by the finite difference method (left) and the finite element method (right).*

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Computer models for two-dimensional steady-state heat conduction

M.R. Albert and G. Phetteplace

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This report outlines the development and verification of two computer models of two-dimensional steady-state heat conduction, including a variety of boundary conditions. One is a finite difference program and the other is a finite element program. The results of each program are compared to two analytic solutions, and to one another. Sample input and output, and user instructions for each program are included. The programs have the same accuracy when modeling problems involving rectangular boundaries. The finite element method is better able to model curved boundaries, while the finite difference program is easier to use.

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PREFACE

This report was prepared by Mary Remley Albert, Mathematician, and Gary E. Phetteplace, Mechanical Engineer, both of the Applied Research Branch, Experimental Engineering Division, U.S. Army Cold Regions Research and Engineering Laboratory. The work was sponsored by DA Project 4A762730AT42, *Design, Construction and Operations Technology for Cold Regions,* Technical Area D, *Cold Regions Design and Construction,* Work Unit 017, *Heat Distribution Systems in Cold Regions.*

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COMPUTER MODELS FOR TWO-DIMENSIONAL STEADY-STATE HEAT CONDUCTION

M.R. Albert and G. Phetteplace

INTRODUCTION

Most major Army facilities are heated by central heat distribution systems. Heat losses from these distribution systems are of interest, as are temperatures of soil surrounding the distribution pipes, for several reasons. If we are to design efficient heat distribution systems we must be able to accurately assess the heat losses in order to determine the optimum trade-off between the cost of insulation and the continuing cost of heat loss. In instances where portions of a system have failed and require replacement, again we need to accurately assess potential heat losses to determine the optimum insulation thickness for the areas that must be replaced. In areas of seasonal frost and permafrost, the temperature distribution in the soil around buried heat carrying pipes is of primary importance. Significant thawing of permafrost can cause loss of support and subsequent pipe settlement and possible pipe breakage. In areas of seasonal frost, piping systems should be buried deep enough to prevent freezeup in the event of a system outage.

Several methods exist for analyzing heat transfer problems of this type. Some of the methods which have been applied to conduction heat transfer include: 1) analytical methods, 2) approximate methods, 3) empirical methods, and 4) numerical methods.

Analytical methods are useful for a limited class of problems for which closed form analytical solutions to the heat conduction equation have been found. They are limited because only the simplest geometries may be treated with corresponding ideal boundary conditions. One such solution exists for a buried uninsulated single pipe in a semi-infinite medium with uniform and constant material properties and boundary conditions; this will be discussed later. Because of the very limited class of problems for which closed form analytical solutions have been found, they have little application to real world engineering problems. However, they can be used to find approximate solutions to actual problems.

Approximate methods are actually an extension of analytical methods. In some instances an exact solution to the heat conduction equation cannot be found, yet by making some reasonable assumptions an approximate solution may become possible. Approximate solutions are limited to certain problems, like analytical solutions, except to a lesser extent. They can provide estimates suitable for applications not requiring precise results.

Empirical methods are based on experimental results instead of on the solution of the governing equation. With knowledge of the form of the governing equation and solution, experimental data can be used to find empirical equations which approximate the physical process. A typical example of this approach is the development of conduction shape factors by electrical analog experiments. Again, empirical methods are limited to cases where empirical equations have been found. These equations are usually only applicable to very specific problems and often cannot be accurately extrapolated to describe similar problems.

Numerical methods have found many applications in heat transfer. In short, a numerical method uses an approximation of the governing differential equation, or its solution, at a number of discrete intervals of a region to find an approximate solution to the heat conduction equation over that discretized region. The enormous advantage of numerical methods is their ability to model nearly any problem, including ones without ideal boundary conditions. This is an advantage no other method can claim. The disadvantage of numberical methods lies in their complexity. Criteria governing the stability, convergence, consistency, and accuracy of a numerical method must be established before the method may be considered valid, and for anything but the simplest problems a computer is required for solutions. Fortunately, however, the computer program can be written to handle a broad class of problems and the user need only be familiar with certain aspects of the problem to accurately use the program to find the results.

The purpose of this report is to outline the finite difference and finite element numerical methods as they apply to steady-state heat conduction, to compare the results of each method to analytical results and also to the results of the other method, and to demonstrate the use of each method's computer program to the potential user.

THE FINITE DIFFERENCE METHOD

Basic ideas of finite differences

The finite difference method is a numerical method used to approximate the solution of a partial differential equation. Common partial differential equations for which finite differences have been employed include the wave equation, the vorticity equation, Poisson's equation, and the heat conduction equation,

$$\frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) = \rho C_p\,\frac{\partial T}{\partial t}\ .$$

Often boundary conditions or nonlinearities in a problem involving differential equations complicate the problem so that analytical solutions are unavailable and numerical solutions must be used. The finite difference method approximates each partial derivative in the differential equation by an algebraic finite difference representation.

Finite difference representations may be derived either from a Taylor series expansion about a point or from physical considerations, such as an energy balance in the case of heat conduction. Let us briefly review the Taylor series procedure. Consider the forward Taylor expansion of a function $f(x)$ about x:

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2}\,f''(x) + \frac{(\Delta x)^3}{6}\,f'''(x) + \ldots \qquad (1)$$

where the primes denote differentiation with respect to x. This may be solved for $f'(x)$ by truncating the higher-order derivatives to obtain

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x)\ . \qquad (2)$$

This is the forward finite difference expression for the first derivative of the function $f(x)$. The term $O(\Delta x)$ indicates that the error due to truncation of the Taylor series is of the order Δx.

The representation is commonly used, for example, as an approximation of $\partial T/\partial t$ in the heat conduction equation, where we let

$$\frac{\partial T}{\partial t} = \frac{T(t + \Delta t) - T(t)}{\Delta t} \tag{3}$$

where

$T(t)$ = temperature at time t
Δt = time step.

Similarly, we may find the backward finite difference expression for the first derivative

$$f(x - \Delta x) = f(x) - (\Delta x)f'(x) + \frac{(\Delta x)^2}{2} f''(x) - \frac{(\Delta x)^3}{6} f'''(x) + \ldots \tag{4}$$

where

$$f'(x) = \frac{f(x) - f(x - \Delta x)}{\Delta x} + O(\Delta x) .$$

To obtain a difference representation for the second derivative, eq 1 and 4 are added

$$f(x + \Delta x) + f(x - \Delta x) = 2f(x) + (\Delta x)^2 f''(x) + O[(\Delta x)^2]$$

$$f''(x) = \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{(\Delta x)^2} . \tag{5}$$

Equation 5 is the central difference representation for the second derivative. Note that it is accurate to the order of $(\Delta x)^2$. It is evident that the accuracy of a solution found using finite differences will be dependent upon our taking Δx sufficiently small.

When using the finite difference method, we set up a network of grid points, called "nodes," in the region of the spatial independent variables in which we are interested. The boundaries of the grid should coincide with those of the problem. The partial derivatives of the original partial differential equation are replaced by their corresponding difference representations, and an equation is set up for each point in the grid. The resulting set of algebraic equations may then be solved.

Finite difference computer program

A finite difference computer program was written to model steady-state two-dimensional heat conduction. The program has been set up in a general form to allow many different conductivities in the region of interest and to permit constant flux, convective, constant temperature, and semi-infinite boundary conditions. The general form of the program allows new problems to be set up very easily.

Application of finite differences to steady-state heat conduction

For two-dimensional steady-state heat conduction, heat transfer obeys the elliptic partial differential equation

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) = 0 \tag{6}$$

where

T = temperature
k = thermal conductivity
x,y = spatial variables.

If the conductivity k were to be constant over the region, the finite difference form for $\partial/\partial x(k\partial T/\partial x) = k\partial^2 T/\partial x^2$ would be immediately obtainable from eq 5,

$$k\frac{\partial^2 T}{\partial x^2} = k\,\frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{(\Delta x)^2}\,.$$

However, the finite difference computer program was to be written so that conductivity could be a function of location. Then the resultant conductivity must be used in regions of varying conductivity. The "resultant conductivity" is easiest to understand by recalling that conductivity is the reciprocal of resistance. The net resistance between any node x, y and node $x+\Delta x, y$ is

$$R = R_x + R_{x+\Delta x} = \frac{d_x}{k_x} + \frac{d_{x+\Delta x}}{k_{x+\Delta x}}$$

where, for example, d_x is the fraction of the distance along the heat flow path which is associated with conductivity k_x. In this program, space increments Δx and Δy are assumed to be uniform, then $d_x = d_{x+\Delta x} = ½$. The resultant conductivity is the reciprocal of resistance R:

$$\frac{1}{R} = \frac{2}{\dfrac{1}{k_x} + \dfrac{1}{k_{x+\Delta x}}}\,.$$

Thus the finite difference formulation for each term in eq 6 is as follows:

$$\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) = \left(\frac{2}{\dfrac{1}{k_{x+\Delta x}} + \dfrac{1}{k_x}}\right)\frac{T(x+\Delta x) - T(x)}{(\Delta x)^2} + \left(\frac{2}{\dfrac{1}{k_{x-\Delta x}} + \dfrac{1}{k_x}}\right)\frac{T(x-\Delta x) - T(x)}{(\Delta x)^2} \qquad (7)$$

$$\frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right) = \left(\frac{2}{\dfrac{1}{k_{y+\Delta y}} + \dfrac{1}{k_y}}\right)\frac{T(y+\Delta y) - T(y)}{(\Delta y)^2} + \left(\frac{2}{\dfrac{1}{k_{y-\Delta y}} + \dfrac{1}{k_y}}\right)\frac{T(y-\Delta y) - T(y)}{(\Delta y)^2}\,. \qquad (8)$$

Combining eq 7 and 8, we arrive at the finite difference formulation for eq 6

$$\left(\frac{2}{\dfrac{1}{k_{x+\Delta y}} + \dfrac{1}{k_x}}\right)\frac{T_{x+\Delta x,y} - T_{x,y}}{(\Delta x)^2} + \left(\frac{2}{\dfrac{1}{k_{x-\Delta x}} + \dfrac{1}{k_x}}\right)\frac{T_{x-\Delta x,y} - T_{x,y}}{(\Delta x)^2} +$$

$$\left(\frac{2}{\dfrac{1}{k_{y+\Delta y}} + \dfrac{1}{k_y}}\right)\frac{T_{x,y+\Delta y} - T_{x,y}}{(\Delta y)^2} + \left(\frac{2}{\dfrac{1}{k_{y-\Delta y}} + \dfrac{1}{k_y}}\right)\frac{T_{x,y-\Delta y} - T_{x,y}}{(\Delta y)^2} = 0\,. \qquad (9)$$

Here it is convenient to change the notation to a more commonly used form. Let $T_{i,j}$ represent the temperature of the node under consideration, where i is increased in the vertical direction and j in the horizontal direction. The grid set up will appear as shown in Figure 1 (the length and width of the grid are variable, and may be established when data are entered into the computer program).

Each node represents the area (enclosed in dotted lines in Fig. 1) *around it*. The most convenient way to set up the computer program to enable it to handle many different cases is to use a square grid, that is, let $\Delta x = \Delta y = \Delta s$. Now we multiply each term in eq 9 by $(\Delta s)^2$, change to the new notation, and rearrange the terms to arrive at the usual finite difference equation for a node i,j not on a boundary,

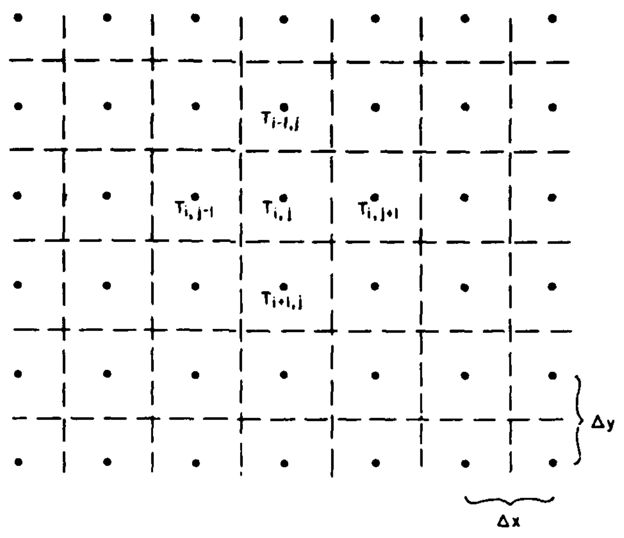


Figure 1. Finite difference grid (boundaries not yet specified).

$$\left(\frac{2}{\frac{1}{k_{i,j-1}}+\frac{1}{k_{i,j}}}\right)T_{i,j-1}+\left(\frac{2}{\frac{1}{k_{i,j+1}}+\frac{1}{k_{i,j}}}\right)T_{i,j+1}+\left(\frac{2}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i,j}}}\right)T_{i+1,j}$$

$$+\left(\frac{2}{\frac{1}{k_{i-1,j}}+\frac{1}{k_{i,j}}}\right)T_{i-1,j}-\left(\frac{2}{\frac{1}{k_{i,j-1}}+\frac{1}{k_{i,j}}}+\frac{2}{\frac{1}{k_{i,j+1}}+\frac{1}{k_{i,j}}}\right.$$

$$\left.+\frac{2}{\frac{1}{k_{i-1,j}}+\frac{1}{k_{i,j}}}+\frac{2}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i,j}}}\right)T_{i,j}=0\,. \qquad (10)$$

Note that each node represents a region of space, and will therefore have a specified thermal conductivity. The above equation allows for a different conductivity for each node of the grid.

Boundary conditions

The finite difference equations used in heat transfer may be derived either by replacement of a partial derivative by its difference representation, as above, or by calculating an energy balance between a node and each of its surrounding nodes. The boundary conditions in the following sections will be derived using energy balance considerations.

Constant flux boundary

For a node on a constant flux boundary, the condition $dT/dx = C$ exists at the boundary. Consider a node on the right-hand boundary. The control volume is that which is associated with the node, as illustrated in Figure 2. Examine the heat flow between node i,j and its surrounding nodes. Between node i,j and node i-1,j we have, for unit depth,

$$Q_1=\left(\frac{2}{\frac{1}{k_{i-1,j}}+\frac{1}{k_{i,j}}}\right)\frac{\Delta x}{2}\;\frac{1}{\Delta y}\;(T_{i-1,j}-T_{i,j})$$

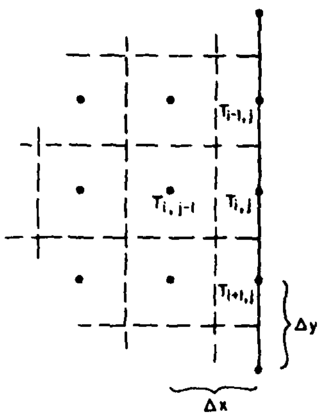


Figure 2. Node on boundary of finite difference grid.

where

$$\frac{2}{\frac{1}{k_{i-1,j}}+\frac{1}{k_{i,j}}}$$ = resultant thermal conductivity between nodes i-1,j and i,j

$$\frac{\Delta x}{2}$$ = surface area perpendicular to the direction of heat flow, for unit depth

Δy = distance between the nodes

$T_{i-1,j} - T_{i,j}$ = temperature difference between nodes.

Similarly, for the heat flow from node i+1,j into node i,j we have

$$Q_3 = \left(\frac{2}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i,j}}}\right) \frac{\Delta x}{2\Delta y} \left(T_{i+1,j} - T_{i,j}\right),$$

and the flow from node i,j-1 into node i,j is

$$Q_2 = \left(\frac{2}{\frac{1}{k_{i,j-1}}+\frac{1}{k_{i,j}}}\right) \frac{\Delta y}{\Delta x} \left(T_{i,j-1} - T_{i,j}\right).$$

The heat flow crossing the boundary into node i,j is given by

$$Q_4 = \phi \cdot \Delta s$$

where ϕ is the heat flux crossing the boundary per unit area. At steady state the sum of the heat flows is zero. Then we have

$$Q_1 + Q_2 + Q_3 + Q_4 = 0.$$

For $\Delta x = \Delta y = \Delta s$,

$$\left(\frac{1}{\frac{1}{k_{i-1,j}}+\frac{1}{k_{i,j}}}\right) T_{i-1,j} + \left(\frac{2}{\frac{1}{k_{i,j-1}}+\frac{1}{k_{i,j}}}\right) T_{i,j-1} + \left(\frac{1}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i+1,j}}}\right) T_{i+1,j}$$

$$-\left(\frac{1}{\frac{1}{k_{i-1,j}}+\frac{1}{k_{i,j}}}+\frac{2}{\frac{1}{k_{i,j-1}}+\frac{1}{k_{i,j}}}+\frac{1}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i,j}}}\right)T_{i,j}=-\phi\Delta s\,. \tag{11}$$

This equation applies to the right-hand boundar\ ; the indices may be appropriately rearranged to obtain the equation for other boundaries. Note that, for a boundary which is insulated or on a line of symmetry, the zero heat flux condition holds and $\phi = 0$.

Convective boundary

Again consider the boundary illustrated in Figure 2. When surface convection is present, the heat flow at the boundary is $Q = h\Delta y(T_A - T_{i,j})$, where T_A is the ambient temperature outside the grid, and h is the surface heat transfer coefficient. The heat flow between node i, j and the surrounding three nodes will be the same as that given for the constant flux boundary. The steady-state formulation for the right-hand boundary with $\Delta x = \Delta y = \Delta s$ is

$$\left(\frac{1}{\frac{1}{k_{i-1,j}}+\frac{1}{k_{i,j}}}\right)(T_{i-1,j}-T_{i,j})+\left(\frac{2}{\frac{1}{k_{i,j-1}}+\frac{1}{k_{i,j}}}\right)(T_{i,j-1}-T_{i,j})$$

$$+\left(\frac{1}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i,j}}}\right)(T_{i+1,j}-T_{i,j})+h\Delta s(T_A-T_{i,j})=0\,. \tag{12}$$

and rearranging gives,

$$\left(\frac{1}{\frac{1}{k_{i-1,j}}+\frac{1}{k_{i,j}}}\right)T_{i-1,j}+\left(\frac{2}{\frac{1}{k_{i,j-1}}+\frac{1}{k_{i,j}}}\right)T_{i,j-1}$$

$$+\left(\frac{1}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i,j}}}\right)T_{i+1,j}-\left(\frac{1}{\frac{1}{k_{i-1,j}}+\frac{1}{k_{i,j}}}+\frac{1}{\frac{1}{k_{i,j-1}}+\frac{1}{k_{i,j}}}\right.$$

$$\left.+\frac{1}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i,j}}}+h\Delta s\right)T_{i,j}=-h\Delta sT_A\,. \tag{13}$$

Again, the indices may be suitably rearranged for nodes on other boundaries.

Constant temperature boundary

For a constant temperature node on any boundary or inside the gride we have $T_{i,j} = C$, where C is the temperature of the node.

Semi-infinite boundary

The semi-infinite boundary condition represents a continuous, uniform material extending ad infinitum in one direction, with a known temperature at infinity. This boundary is approximated in the finite difference computer program by specifying a large distance between the boundary node and the node adjacent to it, and the condition requires a special heat balance for nodes adjacent to the boundary as well as for the boundary nodes. For example, consider the right-hand semi-infinite boundary illustrated in Figure 3, on the edge of a grid with uniform internodal

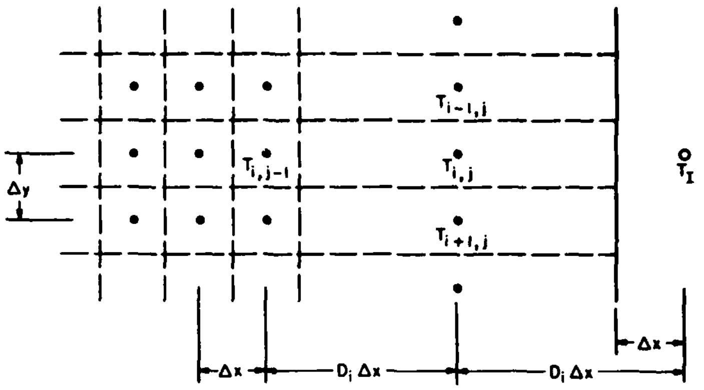


Figure 3. Semi-infinite boundary approximation.

distances. The temperature at infinity is represented by an imaginary node I, located outside the grid. $D_i \cdot \Delta x$ is the distance from the internal node to the node on the boundary, and also the distance from the boundary node to the imaginary node at infinity. As previously stated, the accuracy of the calculated temperature distribution is dependent on the distance between nodes; thus when choosing D_i, we should choose the smallest distance which may still be considered large in terms of the grid size.

The heat flow between node i, j and each of the surrounding nodes is as follows:

$$Q_1 = \left(\frac{2}{\frac{1}{k_{i-1,j}} + \frac{1}{k_{i,j}}}\right) (2D_i - 1)\,\Delta x \cdot \frac{1}{\Delta y} \quad (T_{i-1,j} - T_{i,j}),$$

$$Q_2 = \left(\frac{2D_i}{\frac{1}{k_{i,j-1}} + \frac{2D_i - 1}{k_{i,j}}}\right) \Delta y \cdot \frac{1}{D_i\,\Delta x} \quad (T_{i,j-1} - T_{i,j}),$$

$$Q_3 = \left(\frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}}\right) (2D_i - 1)\,\Delta x \cdot \frac{1}{\Delta y} \quad (T_{i+1,j} - T_{i,j}),$$

$$Q_4 = \left(\frac{2D_i}{\frac{1}{k_I} + \frac{2D_i - 1}{k_{i,j}}}\right) \Delta y \cdot \frac{1}{D_i\,\Delta x} \quad (T_I - T_{i,j})\,.$$

For steady state, the heat balance yields $Q_1 + Q_2 + Q_3 + Q_4 = 0$. Allowing $\Delta x = \Delta y$, we find

$$\left(\frac{2(2D_i-1)}{\frac{1}{k_{i-1,j}} + \frac{1}{k_{i,j}}}\right) T_{i-1,j} + \left(\frac{2}{\frac{1}{k_{i,j-1}} + \frac{2D_i-1}{k_{i,j}}}\right) T_{i,j-1}$$

$$+ \left(\frac{2(2D_i-1)}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}}\right) T_{i+1,j} - \left(\frac{2(2D_i-1)}{\frac{1}{k_{i-1,j}} + \frac{1}{k_{i,j}}} + \frac{2}{\frac{1}{k_{i,j-1}} + \frac{2D_i-1}{k_{i,j}}} + \right.$$

$$+ \frac{2(2D_i-1)}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} + \frac{2}{\frac{1}{k_i} + \frac{2D_i-1}{k_{i,j}}} \Bigg) T_{i,j} = - \frac{2\,T_i}{\frac{1}{k_i} + \frac{2D_i-1}{k_{i,j}}} \; . \qquad (14)$$

Now consider the heat flow for the square node adjacent to the irregular semi-infinite node, as illustrated in Figure 4.

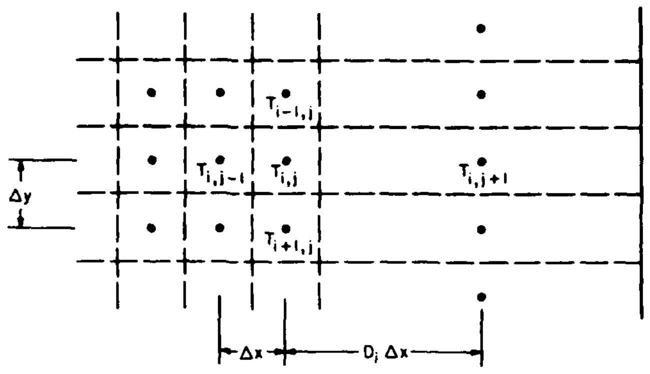


Figure 4. Node adjacent to semi-infinite boundary node.

The heat flow between the node and each of its adjacent nodes is

$$Q_1 = \left(\frac{2}{\frac{1}{k_{i-1,j}} + \frac{1}{k_{i,j}}} \right) \Delta x \cdot \frac{1}{\Delta y} \; (T_{i-1,j} - T_{i,j}),$$

$$Q_2 = \left(\frac{2}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}} \right) \Delta y \cdot \frac{1}{\Delta x} \; (T_{i,j-1} - T_{i,j}),$$

$$Q_3 = \left(\frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} \right) \Delta x \cdot \frac{1}{\Delta y} \; (T_{i+1,j} - T_{i,j}),$$

$$Q_4 = \left(\frac{2\,D_i}{\frac{2D_i-1}{k_{i,j+1}} + \frac{1}{k_{i,j}}} \right) \Delta y \cdot \frac{1}{D_i \Delta x} \; (T_{i,j+1} - T_{i,j}) \, .$$

The resulting equation for steady state, for $\Delta x = \Delta y$, is

$$\left(\frac{2}{\frac{1}{k_{i-1,j}} + \frac{1}{k_{i,j}}} \right) T_{i-1,j} + \left(\frac{2}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}} \right) T_{i,j-1} + \left(\frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} \right) T_{i+1,j}$$

$$+ \left(\frac{2}{\frac{2D_i-1}{k_{i,j+1}} + \frac{1}{k_{i,j}}} \right) T_{i,j+1} - \left(\frac{2}{\frac{1}{k_{i-1,j}} + \frac{1}{k_{i,j}}} + \frac{2}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}} + \right.$$

$$+ \frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} + \left(\frac{2}{\frac{2D_i - 1}{k_{i,j+1}} + \frac{1}{k_{i,j}}} \right) T_{i,j} = 0 \qquad (15)$$

Constant flux corner

Consider the corner shown in Figure 5, subject to the constant flux condition on two sides,

$$Q_1 = \left(\frac{2}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}} \right) \frac{\Delta y}{2\Delta x} (T_{i,j-1} - T_{i,j})$$

$$Q_2 = \left(\frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} \right) \frac{\Delta x}{2\Delta y} (T_{i+1,j} - T_{i,j})$$

$$Q_3 = \tfrac{1}{2} (\phi_T + \phi_S) \Delta s .$$

ϕ_T is the heat flux per unit area crossing the top boundary, ϕ_S is that crossing the side. For steady state, $Q_1 + Q_2 + Q_3 = 0$. In general, for $\Delta x = \Delta y = \Delta s$,

$$\left(\frac{1}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}} \right) T_{i,j-1} + \left(\frac{1}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} \right) T_{i+1,j}$$

$$- \left(\frac{1}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}} + \frac{1}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} \right) T_{i,j} = -(\phi_T + \phi_S) \frac{\Delta s}{2} . \qquad (16)$$

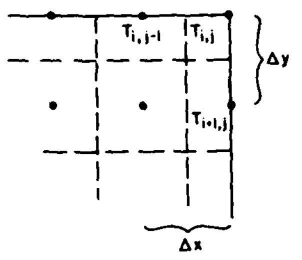


Figure 5. Node on corner of finite difference grid.

Convective corner

For the corner shown in Figure 5, subject to convection on both sides, convection along the top surface yields

$$Q_4 = h \frac{\Delta x}{2} (T_A - T_{i,j}),$$

and along the right side,

$$Q_3 = h \frac{\Delta y}{2} (T_A - T_{i,j}) .$$

With Q_1 and Q_2 the same as for the previous corner, and for $\Delta x = \Delta y = \Delta s$, the equation for steady state is as follows:

$$\left(\frac{1}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}}\right) T_{i,j-1} + \left(\frac{1}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}}\right) T_{i+1,j}$$

$$- \left(\frac{1}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}} + \frac{1}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} + h\Delta s\right) T_{i,j} = -h\Delta s T_A \,. \qquad (17)$$

Semi-infinite corner

Consider the node pictured in Figure 6, on the corner between two semi-infinite boundaries. T_{I_r} is the temperature a large distance to the right of the grid, and T_{I_t} is the temperature away from the top of the grid.

$$Q_1 = \left(\frac{2D_i}{\frac{1}{k_{I_t}} + \frac{2D_i-1}{k_{i,j}}}\right) (2D_i-1)\,\Delta x \cdot \frac{1}{D_i \cdot \Delta y} \left(T_{I_t} - T_{i,j}\right),$$

$$Q_2 = \left(\frac{2D_i}{\frac{1}{k_{i,j-1}} + \frac{2D_i-1}{k_{i,j}}}\right) (2D_i-1)\,\Delta y \cdot \frac{1}{D_i \cdot \Delta x} \left(T_{i,j-1} - T_{i,j}\right),$$

$$Q_3 = \left(\frac{2D_i}{\frac{1}{k_{i+1,j}} + \frac{2D_i-1}{k_{i,j}}}\right) (2D_i-1)\,\Delta x \cdot \frac{1}{D_i \Delta y} \left(T_{i+1,j} - T_{i,j}\right)$$

$$Q_4 = \left(\frac{2D_i}{\frac{1}{k_{I_r}} + \frac{2D_i-1}{k_{i,j}}}\right) (2D_i-1)\,\Delta y\, \frac{1}{D_i \Delta x} \left(T_{I_r} - T_{i,j}\right).$$

For steady state, the heat flows sum to zero. Summing the above, then multiplying each term by $1/(2D_i-1)$ and allowing $\Delta x = \Delta y$, gives us the heat balance for this corner node

$$\left(\frac{2}{\frac{1}{k_{i,j-1}} + \frac{2D_i-1}{k_{i,j}}}\right) T_{i,j-1} + \left(\frac{2}{\frac{1}{k_{i+1,j}} + \frac{2D_i-1}{k_{i,j}}}\right) T_{i+1,j} - \left(\frac{2}{\frac{1}{k_{I_t}} + \frac{2D_i-1}{k_{i,j}}}\right.$$

$$\left. + \frac{2}{\frac{1}{k_{i,j-1}} + \frac{2D_i-1}{k_{i,j}}} + \frac{2}{\frac{1}{k_{i+1,j}} + \frac{2D_i-1}{k_{i,j}}} + \frac{2}{\frac{1}{k_{I_r}} + \frac{2D_i-1}{k_{i,j}}}\right) T_{i,j}$$

$$= -\left(\frac{2}{\frac{1}{k_{I_t}} + \frac{2D_i-1}{k_{i,j}}}\right) T_{I_t} - \left(\frac{2}{\frac{1}{k_{I_r}} + \frac{2D_i-1}{k_{i,j}}}\right) T_{I_r} \,. \qquad (18)$$

In the computer program, it is assumed that $k_{I_t} = k_{i,j} = k_{I_r}$.

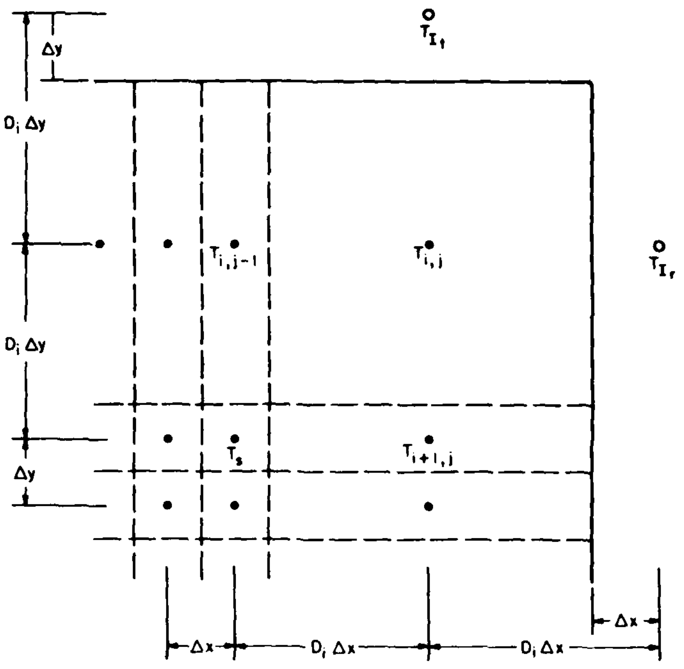


Figure 6. Node on corner between two semi-infinite boundaries.

When this semi-infinite condition is used for a corner, a special heat balance for the interior node labeled T_S in Figure 6 must be used. This balance will be the same as that given by eq 15, except that Q_1 will be replaced by

$$Q_1 = \left(\frac{2D_i}{\frac{2D_i-1}{k_{i-1,j}} + \frac{1}{k_{i,j}}}\right) \Delta x \, \frac{1}{D_i\Delta y} \quad (T_{i-1,j} - T_{i,j}).$$

Then the full heat balance is given by

$$\left(\frac{2}{\frac{2D_i-1}{k_{i-1,j}} + \frac{1}{k_{i,j}}}\right) T_{i-1,j} + \left(\frac{2}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}}\right) T_{i,j-1} + \left(\frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}}\right) T_{i+1,j}$$

$$+ \left(\frac{2}{\frac{2D_i-1}{k_{i,j+1}} + \frac{1}{k_{i,j}}}\right) T_{i,j+1} - \left(\frac{2}{\frac{2D_i-1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} + \frac{2}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}}\right.$$

$$\left. + \frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} + \frac{2}{\frac{2D_i-1}{k_{i,j+1}} + \frac{1}{k_{i,j}}}\right) T_{i,j} = 0 . \qquad (19)$$

Corner with one side constant flux, the other side convective

It is possible to have a corner subject to convection on one side and constant flux on the other side. For the corner in Figure 5, subject to constant flux on the right side and convection on the surface,

$$Q_1 = \left(\frac{2}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}}\right) \frac{\Delta y}{2\Delta x} \quad (T_{i,j-1} - T_{i,j}) \,,$$

$$Q_2 = \left(\frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}}\right) \frac{\Delta x}{2\Delta y} \quad (T_{i+1,j} - T_{i,j}) \,,$$

$$Q_3 = \tfrac{1}{2} \, \phi \Delta s \,,$$

$$Q_4 = h \, \frac{\Delta x}{2} \, (T_A - T_{i,j}) \,.$$

Let $\Delta x = \Delta y = \Delta s$,

$$\left(\frac{1}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}}\right) T_{i,j-1} + \left(\frac{1}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}}\right) T_{i+1,j} - \left(\frac{1}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}} + \frac{1}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} + \tfrac{1}{2} h\Delta s\right) T_{i,j} = - \tfrac{1}{2} h \Delta s T_A - \tfrac{1}{2} \phi \Delta s \,. \qquad (20)$$

Corner with one side constant flux, the other side semi-infinite

As illustrated in Figure 7, let the top right-hand corner of the grid have a semi-infinite boundary on the right side and a constant flux boundary on the top. Allow the heat flow per unit area crossing the top of the grid to be ϕ. Then the heat flows for node i, j are given by

$$Q_1 = \phi(2D_i - 1) \, \Delta x \,,$$

$$Q_2 = \left(\frac{2D_i}{\frac{1}{k_{i,j-1}} + \frac{2D_i - 1}{k_{i,j}}}\right) \left(\frac{\Delta y}{2}\right) \left(\frac{1}{D_i \Delta x}\right) \quad (T_{i,j-1} - T_{i,j}) \,,$$

$$Q_3 = \left(\frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}}\right) \quad (2D_i - 1) \, \Delta x \left(\frac{1}{\Delta y}\right) \quad (T_{i+1,j} - T_{i,j}) \,,$$

$$Q_4 = \left(\frac{2D_i}{\frac{1}{k_I} + \frac{2D_i - 1}{k_{i,j}}}\right) \left(\frac{\Delta y}{2}\right) \left(\frac{1}{D_i \Delta x}\right) (T_I - T_{i,j}) \,.$$

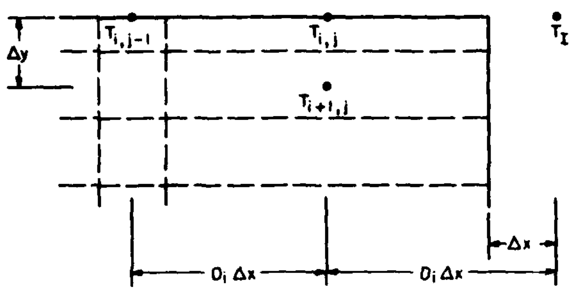


Figure 7. Node on a corner of semi-infinite boundary.

The total heat balance for steady state with $\Delta x = \Delta y = \Delta s$ is given by

$$\left(\frac{1}{\frac{1}{k_{i,j-1}} + \frac{2D_i-1}{k_{i,j}}}\right) T_{i,j-1} + \left(\frac{2(2D_i-1)}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}}\right) T_{i+1,j}$$

$$- \left(\frac{1}{\frac{1}{k_{i,j-1}} + \frac{2D_i-1}{k_{i,j}}} + \frac{2(2D_i-1)}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} + \frac{1}{\frac{1}{k_I} + \frac{2D_i-1}{k_{i,j}}}\right) T_{i,j}$$

$$= -\phi\,(2D_i-1)\,\Delta s - \left(\frac{1}{\frac{1}{k_I} + \frac{2D_i-1}{k_{i,j}}}\right) T_I\,. \tag{21}$$

As is the case with the other semi-infinite boundaries, the node adjacent to the semi-infinite node must be given special consideration. Again, ϕ is the heat flux per unit area crossing the top of the grid.

$$Q_1 = \phi\Delta x,$$

$$Q_2 = \left(\frac{2}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}}\right)\left(\frac{\Delta y}{2}\right)\left(\frac{1}{\Delta x}\right)(T_{i,j-1} - T_{i,j}),$$

$$Q_3 = \left(\frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}}\right)\Delta x\left(\frac{1}{\Delta y}\right)(T_{i+1,j} - T_{i,j}),$$

$$Q_4 = \left(\frac{2D_i}{\frac{2D_i-1}{k_{i,j+1}} + \frac{1}{k_{i,j}}}\right)\left(\frac{\Delta y}{2}\right)\left(\frac{1}{D_i\Delta x}\right)(T_{i,j+1} - T_{i,j})\,.$$

Allow $\Delta x = \Delta y = \Delta s$. The heat balance equation for the node follows:

$$\left(\frac{1}{\frac{1}{k_{i,j-1}}+\frac{1}{k_{i,j}}}\right) T_{i,j-1} + \left(\frac{2}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i,j}}}\right) T_{i+1,j} + \left(\frac{1}{\frac{2D_i-1}{k_{i,j+1}}+\frac{1}{k_{i,j}}}\right) T_{i,j+1}$$

$$-\left(\frac{1}{\frac{1}{k_{i,j-1}}+\frac{1}{k_{i,j}}} + \frac{2}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i,j}}} + \frac{1}{\frac{2D_i-1}{k_{i,j+1}}+\frac{1}{k_{i,j}}}\right) T_{i,j} = -\phi\Delta s\,. \qquad (22)$$

Corner with convection on one side, semi-infinite on the other

Again consider the corner illustrated in Figure 7, but allow convection to occur across the top of the grid. Let h represent the convective heat transfer coefficient, and let T_A be the ambient temperature outside the top of the grid; then the heat flows Q_2, Q_3, and Q_4 will be the same as those given for eq 21 and

$$Q_1 = h(2D_i-1)\,\Delta s\,(T_A - T_{i,j})\,.$$

The heat balance for the corner is given by

$$\left(\frac{1}{\frac{1}{k_{i,j-1}}+\frac{2D_i-1}{k_{i,j}}}\right) T_{i,j-1} + \left(\frac{2(2D_i-1)}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i,j}}}\right) T_{i+1,j} - \left(\frac{1}{\frac{1}{k_{i,j-1}}+\frac{2D_i-1}{k_{i,j}}}\right.$$

$$\left. + \frac{2(2D_i-1)}{\frac{1}{k_{i+1,j}}+\frac{1}{k_{i,j}}} + \frac{1}{\frac{1}{k_1}+\frac{2D_i-1}{k_{i,j}}} + h(2D_i-1)\,\Delta s\right) T_{i,j} = -h(2D_i-1)\,\Delta s\,T_A$$

$$-\left(\frac{1}{\frac{1}{k_1}+\frac{2D_i-1}{k_{i,j}}}\right) T_1\,. \qquad (23)$$

For the node adjacent to the corner node, illustrated in Figure 8, Q_2, Q_3, and Q_4 will be the same as those given for eq 22, and

$$Q_1 = h\Delta s(T_A - T_{i,j})\,.$$

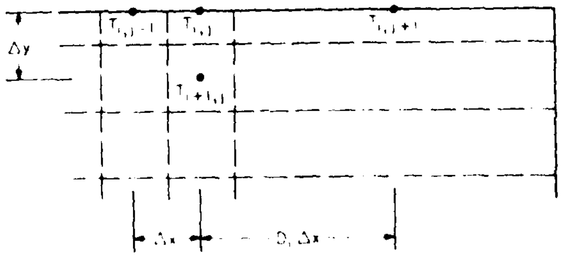


Figure 8. Node adjacent to semi-infinite corner node.

The heat balance for that node is given by

$$\left(\frac{1}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}}\right) T_{i,j-1} + \left(\frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}}\right) T_{i+1,j} + \left(\frac{1}{\frac{2D_r-1}{k_{i,j+1}} + \frac{1}{k_{i,j}}}\right) T_{i,j+1}$$

$$- \left(\frac{1}{\frac{1}{k_{i,j-1}} + \frac{1}{k_{i,j}}} + \frac{2}{\frac{1}{k_{i+1,j}} + \frac{1}{k_{i,j}}} + \frac{1}{\frac{2D_r-1}{k_{i,j+1}} + \frac{1}{k_{i,j}}} + h\Delta s\right) T_{i,j} = -h\Delta s T_A \,. \quad (24)$$

Computer program development

In order to solve a particular problem, a grid whose boundaries coincide with those in the problem must be set up. The number of grid points used is largely a matter of trial and error; as previously stated, the smaller the Δs between the nodes (and hence the more dense the grid), the more accurate the solution. Once the grid is set up, each node in it is assigned the finite difference equation which represents the heat balance between that node and its adjacent nodes. The resulting system of equations may then be solved simultaneously to arrive at the steady-state temperature distribution within the grid.

SSCONDUCT, the program developed to model two-dimensional steady-state heat conduction, is made up of four parts: 1) a data gathering subroutine, 2) the main program, 3) a subroutine to solve the system of equations, and 4) a subroutine to locate the isotherms in the resulting temperature distribution. The grid for the problem is represented by a three-dimensional array, RAY (I, J, K). The first two dimensions designate the spatial location in Cartesian coordinates. The third dimension, K, has three values. RAY (I, J, 1) contains the temperatures for each nodal location I, J. RAY (I, J, 2) describes the location type of each node. Examples of location types include a node on a zero flux boundary, a constant temperature node, a variable interior node, etc. RAY (I, J, 3) is an index of the material type of each node. This index is used to assign conductivities and convection coefficients. It is possible to have a different material at each point in the grid.

Data subroutine

Subroutine SSDATA was written to provide a quick and easy way for the user to set up the grid, without having to worry about formats in the data file. The user has only to edit SSDATA, following directions in the comment statements in the computer program, to insert the desired values of the variables. When SSDATA is run, the new values of the variables are put into the formatted data file STSDAT. The main program then reads the data from STSDAT.

Main program

In the main part of SSCONDUCT, each node of the grid is examined. The conductivities are figured between the node being examined and the adjacent nodes, and the coefficients for the node's difference equation are figured and stored. A grid whose dimensions are X by Y contains XY nodes; hence there will by XY equations with XY unknowns. Since the matrix of coefficients for this system of equations is banded, with all entries being zero outside the band, only the entries of the band need to be stored. The bandwidth is $2X+1$. Thus, instead of storing an XY by XY array for the matrix of coefficients, an XY by 2X+1 array is stored. After each node in the grid has been examined, subroutine BANDMX is called to solve the system of equations.

Subroutine BANDMX

The International Mathematical and Statistical Library's subroutine LEQT1B has been adopted for solving the system of equations which has been stored in band form. Please consult Martin and Williamson (1967) for a detailed discussion of the method.

Subroutine ISOTHM

This subroutine examines the final temperature distribution in the grid, performing a linear interpolation between grid point temperatures to determine the spatial coordinates of user-specified isotherms. The coordinates are written into file POINTS which may then be plotted.

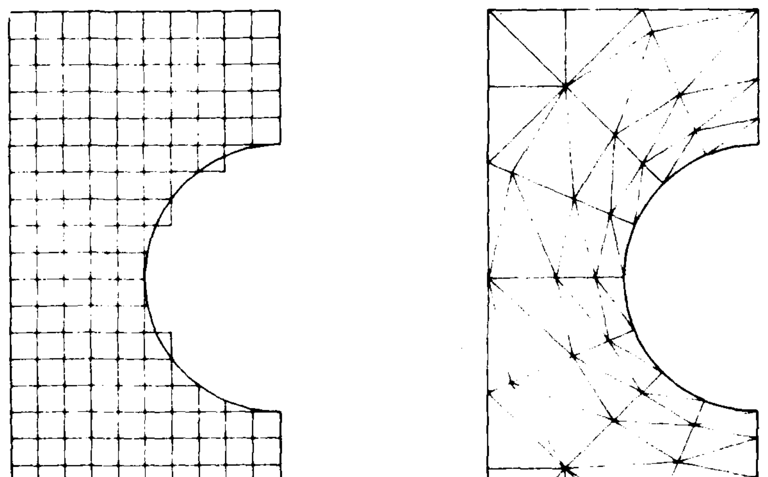
FINITE ELEMENT METHODS

Introduction

Finite element methods are a relatively recent addition to numerical methods. They can be used to obtain approximate solutions to governing differential equations encountered in many disciplines. They were first used by the aerospace industry during the 1950's for structural analysis of complex systems. By the 1960's people found that the method could be applied to a broad class of problems.

Finite element models use a piecewise approximation to the governing differential equations over the region of interest; finite difference models use a pointwise approximation. A major advantage of the finite element approximation over a finite difference approximation is its ability to model irregularly shaped boundaries more accurately. The size of the elements can also be easily varied. The basic shape may also be varied. Figure 9 shows how an irregularly shaped boundary would be modeled with both the finite element and finite difference methods. Each of the models approximates the circular boundary; however, the finite difference model uses a rather crude approximation in comparison. Another major advantage of the finite element method is apparent from Figure 9, that is, we can easily vary the element size. The geometry here is a half symmetry of a buried pipe. We expect the temperature gradients to be much greater in areas nearer the pipe, and by varying element size these areas can be modeled to any degree of accuracy, while surrounding areas with smaller temperature gradients need not be.

The finite element method does have several disadvantages. Since the grid is usually oddly shaped, defining all the coordinates of each node can be a tedious job. Automatic-mesh-generation



a. Finite difference model. *b. Finite element model.*

Figure 9. Finite difference and finite element grids for a circular boundary.

computer programs are available but not necessarily easy to use or inexpensive. Another disadvantage of the finite element method is its complexity. The necessary equations cannot always be entirely based on physical considerations, as can finite difference equations. Finite element computer programs tend to be complex and very hard to modify. In spite of these disadvantages, the finite element method is widely used in many disciplines of engineering today.

Before going into the development of a finite element model for heat conduction, we will first consider the basic steps which must be followed when using the finite element method to solve any problem.

The first step is to discretize the continuum, that is, divide the region to be modeled into elements. Many different shapes can be used for the element. Here we will consider only the most simple two-dimensional element, the triangle. The nodes and elements must now be numbered. As we will see later, this must be done carefully in order to get the most economical solution.

The next step is to select the type of interpolating function which will be used over the element to represent the field variable. Normally, polynomials are chosen since they are easily differentiated and integrated. Since we will be using a simple triangle with only three points our interpolation functions will be linear.

Next we need to find the equations which express the properties of each element. These will be in matrix form. Several approaches are possible. We will rely on the variational principal, which is known for the heat conduction problem. This approach is most convenient, but in some cases a variational principal does not exist for the problem. In these cases, either the direct approach, the energy balance method, or the method of weighted residuals (Galerkin's appraoch) may be used (Huebner 1975).

Now the element equations can be assembled into the global equations describing the entire region. This is a straightforward procedure easily handled by the computer. The system of equations must be modified to account for any boundary conditions present. The result is a system of simultaneous equations.

The next step is to solve this system of equations. This is usually the most time consuming step (computer time). With linear equations, as we will have, many methods exist for solution. The solution is somewhat simplified because the resulting matrix is banded and symmetrical. It is also positive definite. Round-off errors are also reduced because of these properties.

Once we have the solution we may want to compute additional quantities, such as isotherms or temperature gradients in the case of heat conduction.

In the next section we will discuss the procedure used at each of these steps by developing a two-dimensional finite element model for heat conduction. Various types of boundary conditions will be included in this model.

Application to two-dimensional steady-state heat transfer

In this section we will develop a finite element model for heat conduction in two dimensions. The simplest two-dimensional element, the triangle, will be used exclusively. Linear interpolation functions will be used within the elements and provisions for internal heat generation will be included. In an attempt to make the method as clear as possible we will follow the sequence of steps outlined in the last section.

The first step is to divide the region to be modeled into triangular elements. During this step we should try to concentrate the smaller elements in the areas of the highest gradients. Larger elements can be used in areas of smaller gradients. The triangles should be as close to equilateral as possible in order to promote accuracy of the solution. As an example, consider a problem with a geometry similar to that of Figure 9. The region can be divided into triangular elements in a similar fashion as shown in Figure 10.

Now the nodes need to be numbered. Care must be taken here in order to reduce computational requirements. As stated before, the simultaneous equations which result in a finite element

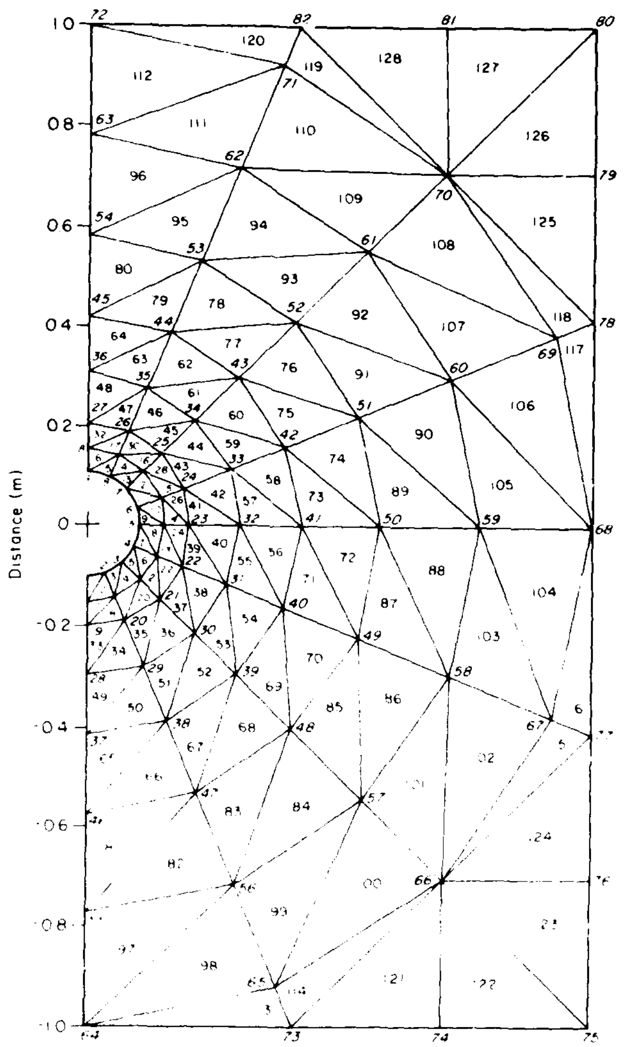


Figure 10. A finite element grid.

equation are banded when represented in matrix form. The smaller the bandwidth, the easier the equations are to solve. Storage requirements are also reduced since only the band need be stored. The numbering of the nodes directly affects the bandwidth of the matrix. The bandwidth is equal to one plus the largest difference between node numbers for any single element (out of all the elements) for a problem with one degree of freedom (temperature) at each node. In Figure 10 the nodes have been numbered in a way that we hope will minimize the bandwidth.

The elements must also be numbered. They may be numbered in any sequence, however, without affecting the computational efficiency of the solution. The computational procedure will require that we know which node numbers are associated with each element. We must also be able to identify what material a particular elements is in, if more than one material exists in the problem. A table in the format of Table 1 will accomplish this purpose.

Table 1. **Typical element data.**

Element number	*Nodes* 1	2	3	*Material type*
1	10	2	1	1
2	10	11	2	1
3	11	3	2	1
4	11	12	3	1
5	12	4	3	1
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

The local node numbers (1, 2, 3 in Table 1) for each element should always be consistently numbered with a particular convention. In this case they are numbered counterclockwise around the element.

We also need to define the spatial coordinates of each of the nodal points. A simple table like Table 2 will accomplish this.

Table 2. Typical nodal point data.

Node	*X*	*Y*
1	0.0000	-0.3280
2	0.1256	-0.3031
3	0.2319	-0.2319
4	0.3031	-0.1286
5	0.3280	0.0000
.	.	.
.	.	.
.	.	.

The coordinates given are for the nodes in Figure 10. The origin in this case is located at the center of the pipe.

Now we need to define the interpolation functions for the elements. Certain continuity requirements must be met by the element type and the interpolation functions, depending on the type of problem being studied. It is beyond the scope of this report to develop these requirements. In this case, the requirements are met by triangular elements and the linear interpolation functions which will be used.

The interpolation function must define the field variable, in this case temperature, over the entire element. That is, given the position of any point within the element, the element interpolation function gives an estimate of the temperature at that point. The form of the linear interpolating polynomial is

$$T = \alpha_1 + \alpha_2 x + \alpha_3 y \tag{25}$$

where T is the temperature, x and y are the coordinates and α_1, α_2 and α_3 are constants.

At local nodes 1, 2 and 3 of the element the following conditions must be satisfied by the interpolating polynomial:

$T = T_1$ at $x = X_1$ and $y = Y_1$

$T = T_2$ at $x = X_2$ and $y = Y_2$

$$T = T_3 \text{ at } x = X_3 \quad \text{and} \quad y = Y_3 . \tag{26}$$

Using each of these conditions and solving for T at each node gives us the following set of equations:

$$T_1 = \alpha_1 + \alpha_2 X_1 + \alpha_3 Y_1$$

$$T_2 = \alpha_1 + \alpha_2 X_2 + \alpha_3 Y_2$$

$$T_3 = \alpha_1 + \alpha_2 X_3 + \alpha_3 Y_3 . \tag{27}$$

Now we can solve this set of equations for the constants α_1, α_2, α_3, yielding

$$\alpha_1 = \frac{1}{2A} \left[(X_2 Y_3 - X_3 Y_2) T_1 + (X_3 Y_1 - X_1 Y_3) T_2 + (X_1 Y_2 - X_2 Y_1) T_3 \right]$$

$$\alpha_2 = \frac{1}{2A} \left[(X_2 Y_3) T_1 + (X_3 Y_1) T_2 + (Y_1 - Y_2) T_3 \right]$$

$$\alpha_3 = \frac{1}{2A} \left[(X_3 X_2) T_1 + (X_1 X_3) T_2 + (X_2 - X_1) T_3 \right] . \tag{28}$$

where A is the area of the triangular element and $2A$ is given by the determinant of the following 3×3 matrix:

$$2A = \begin{vmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{vmatrix} \tag{29}$$

Now we need to find the interpolation function in terms of the temperatures at each node and *the element topography*. We can do this by substituting our expression for the constants α_1, α_2, α_3 into the original interpolation function (eq 25). The result will be an equation for T

$$T^{(e)} = N_1 T_1 + N_2 T_2 + N_3 T_3$$

or

$$T^{(e)} = [N] \{T\}^{(e)} . \tag{30}$$

Here the superscript (e) means "within the element" and N_1, N_2 and N_3 are the shape functions; they are given by the following equations:

$$N_1 = \frac{1}{2A} [a_1 + b_1 x + c_1 y]$$

$$N_2 = \frac{1}{2A} [a_2 + b_2 x + c_2 y]$$

$$N_3 = \frac{1}{2A} [a_3 + b_3 x + c_3 y] . \tag{31}$$

The constants in these equations are given by

$$a_1 = X_2 Y_3 - X_3 Y_2$$

$$a_2 = X_3 Y_1 - X_1 Y_3$$

$$a_3 = X_1 Y_2 - X_2 Y_1$$

$$b_1 = Y_2 - Y_3$$
$$b_2 = Y_3 - Y_1$$
$$b_3 = Y_1 - Y_2$$
$$c_1 = X_3 - X_2$$
$$c_2 = X_1 - X_3$$
$$c_3 = X_2 - X_1 . \tag{32}$$

It is easily shown by substituting back into eq 31 that, for any particular node, the corresponding shape function takes on a value of unity at that node and zero at the other two nodes and the line connecting them.

Now we can use the information in Tables 1 and 2 to assemble the individual element equations into a global system of equations. The result is a set of piecewise continuous equations to approximate temperature over the entire region. Our next step is to relate this to the physical problem of heat conduction in a solid medium.

Several methods are available for finding the element equations, as mentioned earlier. For this problem we will use the variational principal, which is known. A discussion of the calculus of variations is beyond the scope of this report. The interested reader is referred to Pars (1962), Schechter (1967), Huebner (1975), Segerlind (1976) and Zienkiewicz (1977).

The calculus of variations simply states that, if we can find the so-called "functional" which corresponds to the governing differential equation and boundary conditions of the problem, the minimization of that functional requires that its corresponding differential equation and boundary conditions be satisfied. For isotropic two-dimensional steady-state heat transfer, the governing differential equation is

$$\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right) + Q_I = 0 \tag{33}$$

where Q_I is the internal heat generation within the region. Constant temperature boundary conditions are simply

$$T = T_B \quad \text{on} \quad S_1 \tag{34}$$

where

T_B = constant temperature of the boundary S_1
S_1 = segment of boundary at T_B.

Constant heat flux boundary conditions are given by

$$k\,\frac{\partial T}{\partial x}\,\ell_x + k\,\frac{\partial T}{\partial y}\,\ell_y - \phi = 0 \quad \text{on} \quad S_2 \tag{35}$$

where

ℓ_x and ℓ_y = direction cosines of a vector perpendicular to S_2.
S_2 = surface subject to the constant heat flux ϕ.

Finally, the conditions on a convective boundary are

$$k\,\frac{\partial T}{\partial x}\,\ell_x + k\,\frac{\partial T}{\partial y}\,\ell_y + h(T - T_A) = 0 \quad \text{on} \quad S_3 \tag{36}$$

where S_3 is the boundary segment subject to convection.

The entire boundary of the region is made up completely of segments S_1, S_2 and S_3. Radiative boundary conditions are not considered here.

Now we need to use the variational principal to find the element equations. The functional matching eq 33 with boundary conditions (eq 34, 35 and 36) is (Schechter 1967, Huebner 1975, Segerlind 1976)

$$I(T) = \frac{1}{2} \int_A \left[k\left(\frac{\partial T}{\partial x}\right)^2 + k\left(\frac{\partial T}{\partial y}\right)^2 - 2Q_1 T \right] dx\, dy + \int_{S_4} [\phi T + 1/2h(T-T_A)^2]\, dS_4 \tag{37}$$

where S_4 is the union of surfaces S_2 and S_3.

We can define T over the element by recalling eq 30

$$T^{(e)} = \lfloor N \rfloor \{T\}^{(e)} .$$

To minimize the functional $I(T)$, which will in turn satisfy governing differential equation and boundary conditions, we can set its derivative with respect to temperature T equal to zero. Because our interpolation functions meet the continuity requirements as discussed earlier, the total integral $I(T)$ is the sum of the integral $I(T^{(e)})$ for all the elements. If we have M elements,

$$I(T) = \sum_{e=1}^{M} I(T^{(e)}) . \tag{38}$$

Because the temperatures at each node of an element are independent, the partial derivative of the integral $I(T^{(e)})$ with respect to each nodal temperature must be zero:

$$\frac{\partial I(T^{(e)})}{\partial T_1} = \frac{\partial I(T^{(e)})}{\partial T_2} = \frac{\partial I(T^{(e)})}{\partial T_3} = 0 . \tag{39}$$

If node 1 is on the boundary of surface S_4, we have

$$\frac{\partial I(T^{(e)})}{\partial T_1} = 0 = \int_{A^{(e)}} \left[k \frac{\partial T^{(e)}}{\partial x} \frac{\partial}{\partial T_1}\left(\frac{\partial T^{(e)}}{\partial x}\right) + k \frac{\partial T^{(e)}}{\partial y} \frac{\partial}{\partial T_1}\left(\frac{\partial T^{(e)}}{\partial y}\right) \right.$$

$$\left. + Q_1 \frac{\partial T^{(e)}}{\partial T_1} \right] dA^{(e)} + \int_{S_4^{(e)}} \left(\phi \frac{\partial T^{(e)}}{\partial T_1} + h(T-T_A)^{(e)} \frac{\partial T^{(e)}}{\partial T_1} \right) dS_4 . \tag{40}$$

Similar equations apply to other nodes on boundary S_4. For nodes not on boundary S_4, the integral over S_4 is dropped. The derivatives in the above equation can be evaluated by considering eq 30. After substituting these into eq 40, we find the result for node 1 on boundary S_4 is

$$\frac{\partial I(T^{(e)})}{\partial T_1} = 0 = \int_{A^{(e)}} \left[k \left\lfloor \frac{\partial N}{\partial x} \right\rfloor \{T\} \frac{\partial N_1}{\partial x} + k \left\lfloor \frac{\partial N}{\partial y} \right\rfloor \{T\} \frac{\partial N_1}{\partial y} + Q_1 N_1 \right] dA^{(e)}$$

$$+ \int_{S_4} \left[\phi N_1 + h \lfloor N \rfloor N_1 \{T - T_A\} \right] dS_4^{(e)} . \tag{41}$$

The element equation for each element is now found by combining the resulting three nodal equations,

$$\left\{\frac{\partial I}{\partial T^{(e)}}\right\}^{(e)} = \left[\frac{\partial I(T^{(e)})}{\partial T_1}, \frac{\partial I(T^{(e)})}{\partial T_2}, \frac{\partial I(T^{(e)})}{\partial T_3}\right] \quad \text{transpose}$$

$$= [K]^{(e)} \{T\}^{(e)} + \{R\}^{(e)} + [K_{S_4}]^{(e)} \{T\}^{(e)}$$

$$= \left[[K]^{(e)} + [K_{S_4}]^{(e)}\right] \{T\}^{(e)} + \{R\}^{(e)} = 0 . \tag{42}$$

Both the $[K]^{(e)}$ and $[K_{S_4}]^{(e)}$ are 3x3 matrices while $\{R\}^{(e)}$ and $\{T\}^{(e)}$ are 3x 1 vectors. Typical terms within them are

$$K_{12} = \int_{A^{(e)}} k \left(\frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x} + \frac{\partial N_1}{\partial y}\frac{\partial N_2}{\partial y}\right) dA^{(e)}$$

$$R_2 = \int_{A^{(e)}} Q_1 N_2 dA^{(e)} + \int_{S_4^{(e)}} \phi N_2 dS_4^{(e)}$$

$$K_{S_{4_{12}}} = \int_{S_4^{(e)}} h N_1 N_2 dS_4^{(e)} . \tag{43}$$

Now each of these equations for an element can be assembled into the global system of equations. The procedure for doing so is straightforward. If we have n nodes in our problem, our resulting matrix will have dimensions of $n \times n$. Using the data in Table 1 we simply add all of the entries of each of the element matrices into the corresponding global position in the $n \times n$ system matrix. The same basic procedure applies to finding the global R vector.

The assembled system of equations needs to be modified to account for constant temperature boundary conditions. Although exact methods are available for this procedure, we will use an approximate technique which is much simpler to program for computer execution. Before modification, eq 42 is rewritten into the form for solution

$$[K] \{T\} = \{F\} \tag{44}$$

where $\{K\}$ is now the sum of the global $[K]$ and $[K_{S_4}]$ matrices and $\{F\}$ is simply $-\{R\}$. If we have a constant temperature boundary condition at a given node we simply multiply the corresponding diagonal term in the $[K]$ matrix by a relatively large number and also replace the corresponding term in the $\{F\}$ vector by the product of the boundary temperature and this modified diagonal term. In this case we have used 10^{15} as the large number. This is an approximate procedure but it will yield good results as long as the boundary temperature specified is not very small (Segerlind 1976).

Now we are ready to solve for the unknown temperatures. If our problem has n nodes our $[K]$ matrix will be an $n \times n$ matrix and both $\{T\}$ and $\{F\}$ will be $n \times 1$ vectors. It becomes obvious that for large problems, often having hundreds of nodes, this is the most time consuming step. For this reason it is imperative that our solution method be efficient. The fact that our resulting $[K]$ matrix has some special characteristics, as mentioned earlier, greatly simplifies and speeds the solution.

Many techniques are available for solving linear sets of equations such as we have here. The interested reader may refer to one of the many texts on the subject, including Forsythe et al.

(1977) and Gerald (1978). The method used here is in the form of a computer program subroutine. The subroutine is called MCHB and is part of the IBM SSP (Scientific Subroutine Package), which is available on many computer systems. The matrix which is provided to MCHB must be in compressed form, consisting of only the main diagonal and the upper portion of the band, which is symmetrical. The terms are stored in rows in successive storage locations. It returns the temperature, after solution, in the same space that the $\{F\}$ vector was supplied in. The Cholesky decomposition technique is used by MCHB to solve the matrix equations.

All the steps of solution for a two-dimensional steady-state heat transfer problem using the finite element method have now been outlined. The only remaining step is to compute additional quantities based on the resulting temperature distribution. Often, the location of a particular isotherm may be of interest. A simply method of finding the approximate location of an isotherm is simply to interpolate linearly along the element boundary between nodes whose temperatures bound the isotherm of interest.

In the next section we will discuss the computer program based on the theory developed in this section.

Finite element computer program

A finite element computer program, called FEHEAT, has been developed to model two-dimensional steady-state heat conduction. This program is based in part on a program presented by Huebner (1975). The following types of boundary conditions have been included:

1. Constant temperature
2. Specified heat flux (including, of course, zero flux insulated boundaries)
3. Convective heat transfer.

FEHEAT also has the capability to account for internal heat generation.

The program consists of a main program and five subroutines. The main program is responsible for the following operations:

1. Providing dimensioned and common storage space for the necessary matrices, vectors, and scalars
2. Initializing all of the above to zero, as required
3. Determining the size of the problem and type of boundary conditions present
4. Reading in all input data and printing them out for confirmation.
5. Determining the bandwidth of the $[K]$ matrix
6. Calling the appropriate subroutine for solution and isotherm location
7. Printing out resulting temperatures' and isotherms' coordinates

The names and purposes of the five subroutines are:

1. TSM This subroutine forms the $[K]^{(e)}$ matrix (element "thermal stiffness" matrix). A different type of material may be used in each element. TSM computes the constants in the linear interpolation functions as well as the element area. The "thermal stiffness" matrix is modified to account for convection and the internal heat generation at each node is computed here for use in subroutine FRHS.
2. FRHS This subroutine forms the $\{F\}$ vector (right hand side). It begins with the contribution of internal heat generation at the affected nodes and adds to that the effect of specified heat fluxes and convective boundary segments. It is then modified for the constant temperature boundary conditions by the procedures outlined in the previous section.
3. FORMK This subroutine assembles the element $[K]^{(e)}$ matrices to form the global $[K]$ matrix. Constant temperature boundary conditions are accounted for by modification of the corresponding diagonal element as outlined earlier. The resulting matrix is stored, in rows first then in columns, in consecutive storage locations, with only the diagonal and upper portion of the band being stored, as required by MCHB. This procedure greatly reduces storage requirements for banded matrices.

4. MCHB–This subroutine solves the system of equations. The $[K]$ matrix must be stored in the form explained earlier. Double precision is used on several crucial quantities. This subroutine is part of the IBM Scientific Subroutine Package available on many computers.

5. ISOTHM–This subroutine finds the location of any number of specified isotherms at any temperature. It examines each element and performs linear interpolation between adjacent nodes within an element which bounds the temperature of interest. Its results are sets of coordinates for each temperature specified by the user. These results can be plotted with a simple plotting program.

More will be said on the computer program and the preparation of input data in the next section where we examine its application and results for a classical problem. A listing of the computer program, including all the subroutines and sample input files and output data, is given in Appendix B.

PROGRAM VERIFICATIONS AND COMPARISONS

Each computer program was run modeling two steady-state two-dimensional problems: the case of a rectangular plate subject to a uniform temperature on three sides and a sinusoidal temperature distribution across the fourth side, and the case of a buried pipe. Analytical solutions exist for both of these problems. In the *Rectangular plate* section, the rectangular plate problem is solved analytically, and the results are compared to the results of each program. In the *Buried pipe* section, the analytical solution of the buried pipe is presented, along with comparisons of the computer-generated solutions. In the *Semi-infinite condition* section, a one-dimensional problem is solved by the finite difference program to illustrate the use of the semi-infinite boundary.

Rectangular plate

Consider a rectangular plate that has three of its sides at a fixed temperature and has a sinusoidal temperature distribution across the fourth side. The following well-known analytical solution exists to find the steady-state temperature distribution within the plate.

Assume the plate is sufficiently thick so that the end effects may be neglected. The thermal conductivity is constant throughout the plate. We examine a cross section of the plate, and use a shifted temperature $\theta = T - T_0$. The heat flow obeys Laplace's equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 .$$

The boundary condition on the top surface is $\theta = \theta_m \sin \pi x/W$; $\theta = 0$ for the three other surfaces. This gives us the following boundary conditions:

$$\begin{aligned} &(1) \quad \theta\,(0, y) = 0 \\ &(2) \quad \theta\,(W, y) = 0 \\ &(3) \quad \theta\,(x, 0) = 0 \\ &(4) \quad \theta\,(x, H) = \theta_m \sin \frac{\pi x}{W} \; . \end{aligned} \qquad (45)$$

We begin by assuming a solution of the form

$$\theta(x, y) = X(x)Y(y) \; .$$

Then, from Laplace's equation

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} \; .$$

We then proceed as usual with the method of separation of variables, where the separation constant is λ^2:

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \lambda^2, \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \lambda^2$$

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0, \quad \frac{d^2 Y}{dy^2} - \lambda^2 Y = 0$$

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x, \quad Y = C_3 e^{-\lambda y} + C_4 e^{\lambda y}$$

$$\theta = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda y} + C_4 e^{\lambda y}) .$$

Apply the boundary conditions, from eq 45_1,

$$C_1 (C_3 e^{-\lambda y} + C_4 e^{\lambda y}) = 0$$

$$C_1 = 0 .$$

From eq 45_2,

$$(C_2 \cos \lambda x + C_2 \sin \lambda x)(C_3 + C_4) = 0$$

$$C_3 = -C_4 .$$

From eq 45_3, with results of eq 45_1 and 45_2,

$$C_2 C_4 \sin \lambda W (e^{\lambda y} - e^{-\lambda y}) = 0$$

$$2C_2 C_4 \sin \lambda W \sinh \lambda y = 0 .$$

This requires that $\sin \lambda W = 0$ or $\lambda = n\pi/W$ (n is an integer). Because of the linearity of Laplace's equation, θ can be written as a sum of an infinite series

$$\theta = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{W} \sinh \frac{n\pi y}{W}$$

where the constants have been combined.

Now apply the boundary condition from eq 45_4:

$$\theta_m \sin \frac{\pi x}{W} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{W} \sinh \frac{n\pi H}{W} .$$

This holds only if

$$C_1 = \frac{\theta_m}{\sinh \frac{\pi H}{W}}$$

and if $C_2 = C_3 = \ldots = C_n = 0$, then the final temperature distribution in the plate is given by

$$\theta = \theta_m \frac{\sinh \frac{\pi y}{W}}{\sinh \frac{\pi H}{W}} \sin \frac{\pi x}{W}$$

or,

$$T = T_o + T_m \frac{\sinh \frac{\pi y}{W}}{\sinh \frac{\pi H}{W}} \sin \frac{\pi x}{W} . \tag{46}$$

For the computer comparisons, let $T_o = 100°C$, $T_m = 50°C$, $H = 8$ m, and $W = 12$ m. The results were calculated for every 1 m increment of space in the rectangle, and the results are shown in Table 3.

Table 3. The rectangular plate solution.

	*0**	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
0	100.00†	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
1	100.00	100.86	101.66	102.34	102.87	103.20	103.31	103.20	102.87	102.34	101.66	100.86	100.00
2	100.00	101.77	103.43	104.84	105.93	106.62	106.85	106.62	105.93	104.84	103.43	101.77	100.00
3	100.00	102.81	105.43	107.68	109.41	110.49	110.86	110.49	109.41	107.68	105.43	102.81	100.00
4	100.00	104.04	107.81	111.05	113.53	115.09	115.62	115.09	113.53	111.05	107.81	104.04	100.00
5	100.00	105.55	110.73	115.17	118.58	120.73	121.46	120.73	118.58	115.17	110.73	105.55	100.00
6	100.00	107.45	114.39	120.35	124.92	127.80	128.78	127.80	124.92	120.35	114.39	107.45	100.00
7	100.00	109.85	119.04	126.92	132.97	136.78	138.08	136.78	132.97	126.92	119.04	109.85	100.90
8	100.00	112.94	125.00	135.36	143.30	148.30	150.00	148.30	143.30	135.36	125.00	112.94	100.00

*Increments in meters.
†Values in °C.

This problem was run on SSCONDUCT, the finite difference program, using a 17×25 grid (425 nodes) with an internodal spacing of 0.25 m. The results compare to within 0.01°C of the analytical solution.

FEHEAT, the finite element program, also used 425 nodes. The elements were isosceles right triangles with sides 0.25 m long. The total number of elements used was 768. The results from this model predicted the temperature to within 0.02°C of the analytical solution *throughout the region.*

Buried pipe

The solution to the steady-state problem of a buried pipe may be found from vector field theory by superposition of the potentials of two line sources with equal and opposite strength.

In general, the gradient of a scalar field is a vector. In the case of a line source, we assume that the source is of uniform strength all along its length. Then the radial vectors are those which are of interest. In the case of heat flow, allow the radial vectors to be denoted by $\underline{q}$. The scalar potential field is the temperature distribution

$$\underline{q} = -k\underline{\nabla} T \tag{47}$$

where k is the thermal conductivity, and $\underline{q}$ is given by $\underline{q} = W/2\pi r(\underline{\hat{r}})$, where W is the source strength per unit length (W/m), r is the distance from the line source to the point under consideration, and $\underline{r}$ is the unit vector in the radial direction pointing from the line source towards the point under consideration, as shown in Figure 11.

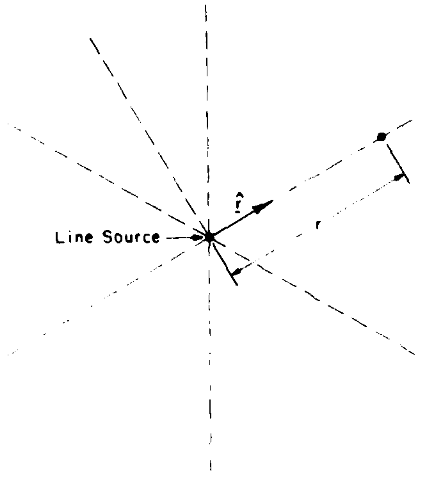


Figure 11. Simple line source.

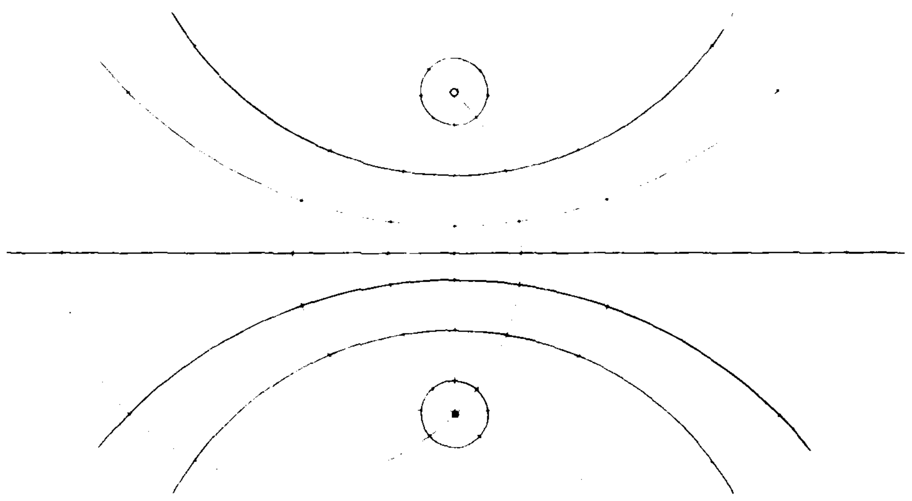


Figure 12. Superposition of two line sources (●–heat source below the ground; ○–heat source above the ground).

Upon substitution into eq 47:

$$\frac{W'}{2\pi r}\,\underline{\hat{r}} = -k\,\frac{\partial T}{\partial r}\,\underline{\hat{r}}$$

$$\int -\frac{W'}{2\pi k r}\,\underline{\hat{r}}\,dr = \int dT\,\underline{\hat{r}}$$

$$-\frac{W'}{2\pi k}\ln\frac{r}{r_0} = T - T_0 = \phi\,. \tag{48}$$

Now consider the situation illustrated in Figure 12. W is the heat source strength below the ground; we also imagine a source of equal and opposite strength an equal distance above the ground. The field lines are the dotted lines, the isotherms are represented by continuous lines. For the bottom source

$$\phi_1 = -\frac{W}{2\pi k} \ln \frac{r_1}{r_o}$$

and, for the top source,

$$\phi_2 = \frac{W}{2\pi k} \ln \frac{r_2}{r_o} .$$

The resultant field (temperature distribution) may be obtained by superposing the potentials from these two fields,

$$\phi_1 + \phi_2 = \frac{W}{2\pi k} \left(\ln \frac{r_2}{r_o} - \ln \frac{r_1}{r_o}\right)$$

$$\phi = \frac{W}{2\pi k} \ln \frac{r_2}{r_1}$$

$$T - T_o = \frac{W}{2\pi k} \ln \frac{r_2}{r_1} . \qquad (49)$$

Isotherms are lines of constant ϕ. When $r_2 = r_1$, the straight line between the two fields is given by

$$\phi = \frac{W}{2\pi k} \ln 1 = 0$$

$$T - T_o = 0$$

$$T = T_o .$$

Thus the reference temperature is given at the ground surface.

Now put the problem on a system of Cartesian coordinates, as illustrated in Figure 13.

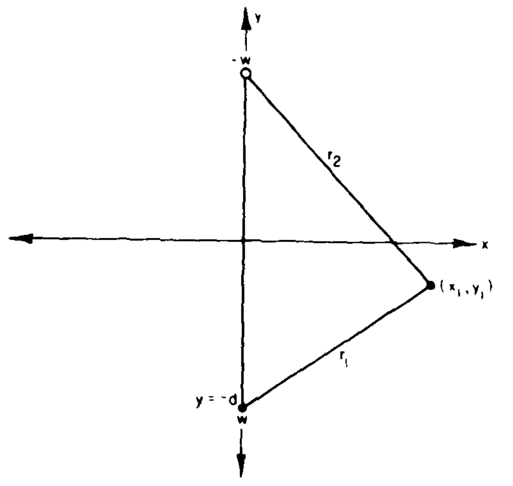


Figure 13. Cartesian coordinates for source problem.

The source of strength W is located at the point $(0, -d)$. From Figure 13, the distances r_2 and r_1 are given by

$$r_2 = \sqrt{x_1^2 + (d-y_1)^2}$$

$$r_1 = \sqrt{x_1^2 + (d+y_1)^2} \ .$$

Substitute these results into eq 49

$$T - T_o = \frac{W'}{2\pi k} \ \ln \frac{\sqrt{x_1^2 + (d-y_1)^2}}{\sqrt{x_1^2 + (d+y_1)^2}}$$

$$\frac{4\pi k(T-T_o)}{W'} = \ln \frac{x_1^2 + (d-y_1)^2}{x_1^2 + (d+y_1)^2}$$

$$\exp\left(\frac{4\pi k}{W'} \ (T-T_o)\right) = \frac{x_1^2 + (d-y_1)^2}{x_1^2 + (d+y_1)^2} \ . \qquad (50)$$

This gives a general equation for the temperature at any point, given the ground surface temperature T_o, the conductivity, and the source strength. It will be shown below that the resulting isotherms have a circular shape. We will let the pipe be represented by one of the isotherms. Given the temperature of that isotherm, we may calculate the resulting temperature distribution, with the temperature of the pipe and the temperature of the ground surface also given.

For a given isotherm, the left-hand side of eq 50 is constant. Let

$$c = \exp\left(\frac{4\pi k}{W'} \ (T-T_o)\right)$$

then

$$c = \frac{x_1^2 + (d-y_1)^2}{x_1^2 + (d+y_1)^2} \ .$$

After the fraction is cleared in this equation, we complete the square to arrive at

$$x_1^2 + \left[y_1 + d\left(\frac{c+1}{c-1}\right)\right]^2 = d^2\left[\left(\frac{c+1}{c-1}\right)^2 - 1\right] .$$

Thus each isotherm is a circle with center c and radius r, as given below:

$$c = (o, d)\left(\frac{c+l}{c-1}\right)$$

$$r = 2d \ \frac{\sqrt{c}}{c-1} \ .$$

For comparison with the computer models, consider the problem of a pipe with a 0.1-m radius at 100°C buried at a depth of 1 m (center of pipe); the ground surface is at 10°C.

The pipe is the 100°C isotherm whose center is at $1 = d\,(c_o + 1/c_o - 1)$

$$r = 0.1 = 2d \ \frac{\sqrt{c_o}}{c_o - 1}$$

Substitute for d to obtain

$$0.1 = 2\left(\frac{c_o - 1}{c_o + 1}\right) \frac{\sqrt{c_o}}{(c_o - 1)}$$

$$c_o^2 - 398\,c_o + 1 = 0$$

$$c_o = 397.9975$$

$$d = \frac{c_o - 1}{c_o + 1} = 0.995\,.$$

Then

$$397.9975 = \exp[4\pi k/W(100-10)]$$

$$5.986 = 360\,\pi k/W$$

$$W = 360\,\pi k/5.986\,.$$

For any isotherm,

$$C = \exp[4\pi k(T - T_o/W)] = \exp[4\pi k(T - T_o)\,(5.986)/360\,\pi k]$$

$$C = \exp[0.0665\,(T - T_o)]\,.$$

Thus, given a point (x, y) of interest, its temperature may be calculated from the following equation:

$$e^{0.0665(T-10)} = \frac{x_1^2 + (0.995 - y_1)^2}{x_1^2 + (0.995 + y_1)^2}$$

$$T = 10 + 15.03396 \ln \frac{x_1^2 + (0.995 - y_1)^2}{x_1^2 + (0.995 + y_1)^2}\,. \qquad (51)$$

The region modeled by the computer programs was a rectangular area extending from the ground surface to 3 m below the surface, and 2 m to each side of the pipe. Because there is a vertical line of symmetry extending from the ground surface down through the center of the pipe, the computer programs modeled only half of the area, and used a zero flux boundary along the right side, which passes through the center of the pipe.

The finite difference program used a 31x 21 grid (651 nodes), with an internodal distance of 0.1 meters. The pipe was represented by four constant temperature nodes. The left side and bottom boundaries were assigned the semi-infinite boundary condition. See Appendix A for the input and output for the program; the grid is labeled and printed in the output file, STDOUT. This problem required 88 CPU-seconds (Central Processing Unit) on CRREL's PRIME 400 computer, a cost of $3.08.

The finite element program used 104 nodes (166 elements). The elements were triangles with sides of varying lengths, as shown in Figure 14. The left side and bottom boundaries were assigned a constant temperature boundary condition. The finite element problem required 26 CPU-seconds on the same computer, and cost $0.91.

The three solutions–analytical, finite difference, and finite element are graphed in Figure 15 for the 30°, 50° and 70°C isotherms. Also shown are the finite difference and finite element 100°C nodes, which represent the pipe. Note that the finite element method is better able to

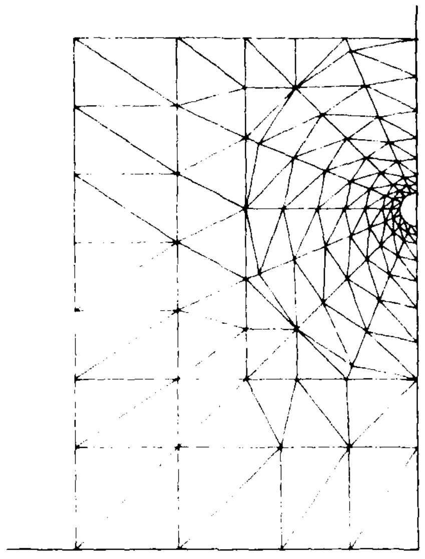


Figure 14. Finite element grid for buried pipe problem.

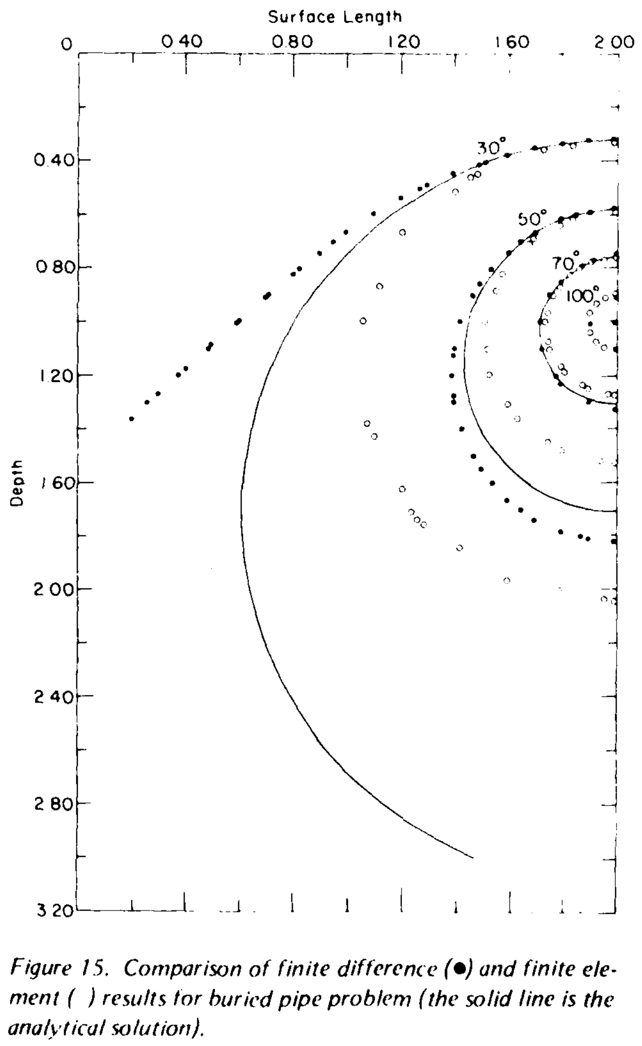


Figure 15. Comparison of finite difference (●) and finite element ( ) results for buried pipe problem (the solid line is the analytical solution).

model the circular surface of the pipe, while the finite difference method used only a four-node representation. Both methods represent the temperatures above the pipe very well, but the finite difference method is slightly more accurate. For regions to the left of the pipe and below the pipe, there are inaccuracies in both of the computed solutions, mainly due to the effect of the semi-infinite boundaries. The finite difference solution overpredicts the distance of the various isotherms from the pipe, but is the more accurate solution in this case. The finite element method underpredicts the isotherm locations. The internodal spacing used could be reduced in both cases should a more accurate solution be necessary.

The semi-infinite condition

The semi-infinite boundary condition in the finite difference program is modeled by use of a long rectangular node at the boundary. The temperature at infinity is known. To verify this approach, consider the one-dimensional problem of a semi-infinite plane. The temperature at the edge is 115°C, and the temperature at infinity is 0°C. If we allow "infinity" to be a large but finite distance, then a linear temperature distribution between 0° and 115°C would be expected at steady state.

A grid 17 nodes long was used to model this situation (Fig. 16). There are 16 nodes of regular shape in the grid, and one semi-infinite boundary node. Let *DS* be the internodal distance for the interior of the grid, and 100·*DS* be the distance from the last square node to the node at infinity. Listed in Table 4 are the expected temperatures for each node and those calculated by SSCONDUCT.

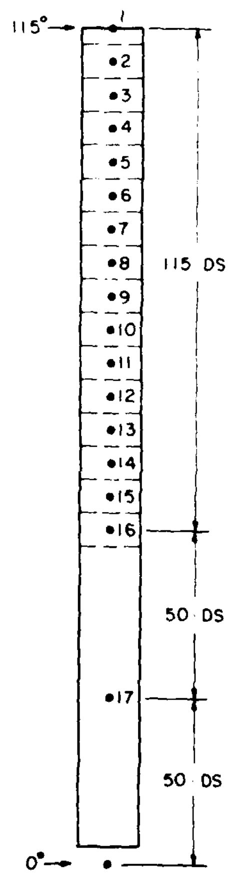


Figure 16. Semi-infinite verification problem.

Table 4. One-dimensional semi-infinite comparison.

Node	*Temperature expected (°C)*	*Temperature calculated (°C)*
1	115	115.00
2	114	114.00
3	113	113.00
4	112	111.99
5	111	110.99
6	110	109.99
7	109	108.99
8	108	107.98
9	107	106.98
10	106	105.98
11	105	104.98
12	104	103.97
13	103	102.97
14	102	101.97
15	101	100.97
16	100	99.97
17	50	49.88

The error at node 17 is small, considering that the distance modeled by the one rectangular node is almost seven times the size of the rest of the grid. Errors encountered when using the semi-infinite condition in two dimensions, however, may be much larger, as may be seen from the buried pipe example. In general the semi-infinite condition should be used in two-dimensional problems only in regions where the temperature gradient is small and precise knowledge of the temperature distribution is not critical.

INSTRUCTIONS FOR USE OF COMPUTER PROGRAMS

This section is provided for those who wish to use either one of the two heat conduction programs. SSCONDUCT is the finite difference program, and FEHEAT is the finite element program.

Instructions for SSCONDUCT

SSCONDUCT was set up to be easily modified to handle new problems. Running a new problem simply involves editing subroutine SSDATA by following instructions provided by comment statements in the program to adjust the variables and arrays. The variables are defined at the beginning of SSDATA.

However, before attempting to alter the data subroutine, the user should make a drawing of the problem. For a problem with more than one thermal conductivity, a number should be assigned to each different material in the problem, starting with 1. Let A be the number of different materials in the problem. Assign the number A to the material which appears the most in the problem. For each material, obtain the thermal conductivity in W/m K. If a material is located on a convective boundary, obtain the convection coefficient in W/m^2 K.

Now the finite difference grid for the problem should be drawn. The boundaries of the grid should match with those in the problem as closely as possible. Except for semi-infinite boundaries, the nodes should be equally spaced. At this point, a decision must be made on a reasonable nodal spacing. To date, there is no general way of determining the optimum nodal spacing, but there are the following guidelines:

1. The nodes should represent the materials in the grid as accurately as possible, i.e. no node should be composed partly of material 1 and partly of material 2.
2. Regions with a steep temperature gradient are best represented by a dense grid.
3. In general, accuracy decreases as internodal spacing increases.
4. The more nodes used, the more computer storage space is required, and the more computer time will be needed to solve the problem.

Bear in mind that the use of semi-infinite nodes introduces an error that is proportional to the area of the semi-infinite node. Therefore, the semi-infinite condition should be used only in regions of small temperature gradients when knowledge of the temperature distribution is not critical.

Let y be the number of rows in the grid, and let x equal the number of columns in the grid. The grid is represented in SSCONDUCT by RAY (I, J, K), where I is increased from 1 to y, J from 1 to x, and K from 1 to 3.

It may be desirable to make three drawings of the grid. The first time, label each node with temperatures where they are known. This will represent RAY (I, J, 1). The second time, label each node with its location type, as defined at the beginning of subroutine SSDATA. For example, if the top left-hand corner of the grid is at constant temperature, label it "11" (RAY [1, 1, 2] = 11). The grid labeled in this fashion represents RAY (I, J, 2). These labels are used in the main program to assign the proper equation to each node. The third time the grid is drawn, label each node with its material number, as determined above. This will represent RAY (I, J, 3).

A copy of the program, including SSDATA, is listed in Appendix A. Editing SSDATA to agree with the new problem will set up the data file when SSDATA is run.

When locating isotherms in the final temperature distribution, subroutine ISOTHM assumes a uniform internodal spacing. Thus if *semi-infinite* boundaries are used, edit ISOTHM as directed in the comment statements at the beginning of the subroutine to avoid error in interpolations.

When SSCONDUCT is run, it asks two questions of the user. The first is whether or not subroutine SSDATA should be run (it should be run if the formatted data file, STSDAT, is empty or if the user desires a change in that file). The second question is whether or not the user wants subroutine ISOTHM to be called to locate isotherms in the final temperature distribution. File STDOUT is the output file which lists the initial data and boundary conditions along with the final temperature distribution.

At the time of publication of this report, all boundary conditions are operational on all sides of the grid, except for the semi-infinite conditions, which are operational on the left edge, bottom, and right edge of the grid only.

Instructions for FEHEAT

The first step in using program FEHEAT is to divide the region to be modeled into triangular elements. Care must be taken here in order to obtain an accurate and economical solution. Points which will help are:

1. Use the smallest elements in the areas where you would expect the highest temperature gradients.

2. Use large elements where small gradients are suspected in order to minimize the number of elements and minimize computer core requirements and running time (i.e. cost).

3. Avoid the use of triangles with high aspect ratios, that is, triangles with sides varying greatly in length.

4. When several materials are present, approximate their boundaries as closely as possible with the element boundaries. Each element can only consist of one material in the model.

The next step is to assign numbers to all of the nodes. Great care must be taken here to ensure computational efficiency. The form of the matrix of equations which yields the solution is directly affected by the numbering of the nodes. The smaller the bandwidth of this matrix, the lower the computational effort required to solve it. To minimize the bandwidth, number the nodes such that, for the entire region, the maximum difference between any two node numbers of any element is as small as possible. For instance, it's better to have a maximum difference of 10 among the node numbers of several elements than to have one element be 15 greater than the smallest element and have a less than 10 difference among the remaining elements. The element or elements with the largest difference between node numbers determines the bandwidth which must be carried for the entire solution.

Once the nodes have been numbered the elements themselves may be numbered. They may be numbered in any sequence without affecting computation efficiency. Now we can construct the input data files. They are

ED	=	element data file
NPD	=	nodal point data file
BCT	=	constant temperature boundary conditions file
IHG	=	internal heat generation file
SHF	=	specified heat flux file
CBS	=	convective boundary segments file
QUAN	=	control data file
TIOT	=	desired isotherm temperatures.

Of these eight input data files only ED, NPD and QUAN are necessary unless the boundary conditions require the use of BCT, IHG, SHF or CBS and the desire for isotherms requires the use of TIOT.

The input data for ED, the element data file, is shown for a sample problem in Table 1. The node numbers for a particular element must be entered in counterclockwise order around the element. The material type must be assigned to each element and the material thermal conductivity defined in the program. Any consistent set of units may be used for the thermal conductivity, temperature, and position variables. The numbers in this file must be in the format (I5, 4I6).

NPD, the nodal point data file, is constructed exactly as shown in Table 2. The X and Y coordinates of each node must be determined from the drawing of the region. These are entered in the format (I5, 2F10.4).

Constant temperature boundary conditions are input in file BCT. The format of this file is (boundary condition number, node number, assigned temperature). These may be listed in any order of node numbers, although the boundary condition numbers must be consecutive. A typical BCT file might appear as:

```
1    3    50.0000
2    7   100.0000
3   10   125.0000
4    2   150.0000
5   39    20.0000
6    8    75.0000
.    .       .
.    .       .
.    .       .
```

Thus, for instance, boundary condition 3 would be on node 10 and it would have an assigned temperature of 125. Again, the units of temperature need only be consistent with their use elsewhere. Headings on the printout indicate temperatures are in °F, but this may be ignored if °C are used for input throughout. The format for input in file BCT is (I5, I6, F10.4).

File IHG contains the boundary conditions for internal heat generation. The format of this file is (boundary condition number, element number, assigned heat generation rate). Similar to file BCT, these may be listed in any order of element numbers, but the boundary condition numbers must be consecutive. A typical file could appear as:

```
1    5    20.0000
2   11    10.0000
3    6    17.5000
4   22     3.3333
5   16    20.0000
6    4    35.0000
.    .       .
.    .       .
.    .       .
```

For instance, boundary condition 5 would be on element 16 where the rate of internal heat generation would be 20. Consistent units must be used for heat generation rate. The format for the input in file IHG is identical to that for file BCT; it is (I5, I6, F10.4).

File SHF contains the boundary conditions of specified heat flux. An ideally insulated boundary has a heat flux of zero. If no heat flux is specified for a node on a physical boundary of the system, it becomes a zero heat flux boundary, provided, of course, that no other boundary condition is specified there. The format of this file is (boundary condition number, node number, specified heat flux). Again, the nodes may be listed in any order but the boundary condition numbers must be consecutive. Consistent units must be used. A typical SHF file is:

```
1    6     5.0000
2   17    27.0000
3   24     6.6667
4    4    32.5000
5    5    50.0000
```

```
6   87    9.5000
.    .    .
.    .    .
.    .    .
```

For example, boundary condition 2 would be on node 17 where the specified heat flux would be 27. Again, the format for input into file SHF is (I5, I6, F10.4).

File CBS contains boundary conditions for segments of the boundary where convection occurs. A boundary segment subject to convection is defined by the nodes at its end. The format of this file is (*boundary condition number, node number at one end, node number at the other end, convective heat transfer coefficient, ambient temperature*). The boundary segments may be listed in any order but the boundary condition number must be consecutive. Consistent units must be used for the convective heat coefficient and the ambient temperature. A typical CBS file is:

```
1    6    7   1.0000   20.0000
2   27    3   1.2500   25.0000
3   12   20   2.3750   55.2750
4   21   20   1.0000   20.0000
.    .    .     .        .
.    .    .     .        .
.    .    .     .        .
```

In this case, for example, convective boundary segment 3 would have node 12 at one end and node 20 at the other. The convective heat transfer coefficient there would be 2.375 and the ambient temperature 55.275. The format for file CBS is (3I5, 2F8.4).

The QUAN file contains control data for the problem. The format of the file is (NN, NE, MC, MJC, MC1, MC2, NIT), where

NN = total number of nodes
NE = total number of elements
MC = total number of fixed temperature boundary conditions
MJC = total number of elements with internal heat generation
MC1 = total number of nodes with specified heat flux
MC2 = total number of boundary segments subject to convection
NIT = number of isotherms desired.

A typical QUAN file is:

```
104   166   26   0   0   0   6.
```

Thus, this problem would have 104 nodes, 166 elements and 26 constant temperature boundary conditions, and 6 isotherms would be desired. The format of this file is (7I6).

The final file used is TIOT. This file contains the temperature of the isotherms which the user wishes to locate. The format is (IT_1, IT_2, IT_3, IT_4..., IT_{NIT}), where IT_i is the temperature of the *i*th isotherm desired, $1 \leq i \leq NIT$. A typical TIOT file is:

```
10.00
30.00
50.00
70.00
90.00
100.00
```

The format for file TIOT is (F8.2)

With the necessary files ready, program FEHEAT can be executed. The output is self explanatory, giving the resultant temperatures at each node. If isotherms are requested, a set of coordinate pairs which will form the isothermal line will also be output. Input data are printed out along with results as a means of checking to see that all data are properly read in.

If problems occur in the use of this program two likely areas should be checked first. These are:

1. All dimension and common statements must be made large enough for the problem. Currently, the dimensions are set for a maximum of 175 nodes, 300 elements and a bandwidth of 30.

2. Most often, mistakes are made in the input data, particularly in files ED and NPD. A very easy way to check these files is to write a short plotting program which would read first the coordinates of all the nodes from the NPD file and store them. Then the program would read in the nodes for each of the elements from the ED file and, using the coordinates of these nodes as obtained from NPD, plot each element individually on a composite plot. After all the elements are plotted it should be exactly as the original discretization of the region was. If any stray lines are found, element sides missing, or incorrect boundaries drawn, the problem in the input data should be easily located.

CONCLUSIONS

Two computer programs have been developed to solve the steady-state heat conduction equation under a variety of boundary conditions. The accuracy is the same for each when modeling problems with rectangular boundaries and no semi-infinite boundary conditions are modeled.

SSCONDUCT, the finite difference program, is by far the ea :est to set up and run for a new problem.

FEHEAT, the finite element program, has the advantage of being able to use elements of varying size throughout the problem. This provides for a better definition of curved boundaries, and makes the problem cheaper to run (less computer time) if large elements are used where the temperature gradient is small.

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APPENDIX A: FINITE DIFFERENCE PROGRAM—DOCUMENTATION AND SAMPLE INPUT AND OUTPUT

Program SSCONDUCT

```
C       SSCONDUCT
C     THIS FORTRAN PROGRAM SOLVES FOR STEADY STATE 2-D TEMPERATURE DISTRIBUTION
C    RESULTING FROM CONDUCTION HEAT TRANSFER. BOUNDARY CONDITIONS MAY INCLUDE
C    CONSTANT TEMPERATURES, CONVECTIVE SURFACES, SEMI-INFINITE, AND CONSTANT
C    FLUX BOUNDARIES. THE GRID MAY CONTAIN MANY DIFFERENT CONDUCTIVITIES.

C    DATA FOR SSCONDUCT IS GATHERED BY SSDATA, WHICH PUTS IT INTO FILE STSDAT.
C    SEE SSDATA FOR AN EXPLANATION OF VARIABLES.
C     FOR DIMENSIONS: RMAT(Y*X,2X+1),RVCT(Y*X,1),RXL(Y*X,(X+1))
C
        IMPLICIT INTEGER(A,B,G,W,X,Y,Z)
        COMMON/M1I/ A,X,Y,NISO
        COMMON/M1R/ DS,DI,TMAX,TMIN,FI(4),THK(2),H(2),TISO(9),
      & RAY(100,100,3)
        COMMON/M3/ RMAT(10004,205),RVCT(10004,5),RXL(10004,105)
        DIMENSION IFLAG(4)
        CALL CONTRL(3,'STSDAT',5)
        CALL CONTRL(2,'STDOUT',6)
        WRITE(1,1)
 1      FORMAT(1X,'IF YOU EDITED THE DATA SUBROUTINE SINCE YOU LAST ',
      C ' RAN THIS, TYPE "1"',/,1X,'AND SSDATA WILL BE EXECUTED.',/,1X,
      C 'OTHERWISE, TYPE "2"')
        READ(1,*,ERR=1111) IT
        GO TO 2
 1111   CONTINUE
        WRITE(1,3)
 3      FORMAT(1X,'ERROR ON INPUT FOR YOUR RESPONSE')
        GO TO 1
 2      CONTINUE
        IF(IT.EQ.2) GO TO 4
        CALL SSDATA
        GO TO 6
 4      CONTINUE
        READ(5,10) DS,DI
 10     FORMAT(1X,2F9.4)
        READ(5,20) A,X,Y,NISO
 20     FORMAT(1X,4I5)
        READ(5,30) (THK(L),L=1,A)
        READ(5,30) (H(L),L=1,A)
        READ(5,30) (TISO(B),B=1,NISO)
 30     FORMAT(1X,F9.4)
        READ(5,31) (FI(I),I=1,4)
 31     FORMAT(1X,4F9.4)
        READ(5,35) ((RAY(I,J,3),J=1,X),I=1,Y)
        READ(5,35) ((RAY(I,J,2),J=1,X),I=1,Y)
        READ(5,35) ((RAY(I,J,1),J=1,X),I=1,Y)
 35     FORMAT(1X,11F7.2)
 6      CONTINUE
C     WRITE DATA FOR RUN INTO STDOUT ***** **** ***
        WRITE(6,50)
        WRITE(1,50)
 50     FORMAT(1X,'    DATA FOR THIS RUN OF SSCONDUCT:',/,/ )
        WRITE(6,51) X
        WRITE(1,51) X
 51     FORMAT(1X,'X=',I3)
        WRITE(6,52) Y
        WRITE(1,52) Y
 52     FORMAT(1X,'Y=',I3)
        WRITE(6,54) A
        WRITE(1,54) A
 54     FORMAT(1X,'A=',I3)
        WRITE(6,56) NISO
        WRITE(1,56) NISO
 56     FORMAT(1X,'NISO=',I3)
        WRITE(6,8) DS
        WRITE(1,8) DS
 8      FORMAT(1X,'DS=',F7.3)
```

```
      WRITE(6,58) DI
      WRITE(1,58) DI
 58   FORMAT(1X,'DI=',F9.4)
      WRITE(6,53)
      WRITE(1,53)
 53   FORMAT(1X,'TISO(B),B=1,NISO:')
      WRITE(6,59) (TISO(B),B=1,NISO)
      WRITE(1,59) (TISO(B),B=1,NISO)
 59   FORMAT(1X,F7.2)
      WRITE(6,55)
      WRITE(1,55)
 55   FORMAT(1X,'THK(L),L=1,A:')
      WRITE(6,30) (THK(L),L=1,A)
      WRITE(1,30) (THK(L),L=1,A)
      WRITE(6,57)
      WRITE(1,57)
 57   FORMAT(1X,'H(L),L=1,A:')
      WRITE(6,30) (H(L),L=1,A)
      WRITE(1,30) (H(L),L=1,A)
      DO 15 I=1,4
      WRITE(6,16) I,FI(I)
      WRITE(1,16) I,FI(I)
 16   FORMAT(1X,'FI(',I1,')=',F9.4)
 15   CONTINUE
      WRITE(6,65)
      WRITE(6,890) ((RAY(I,J,3),J=1,17),I=1,Y)
      WRITE(6,892)
      WRITE(6,893) ((RAY(I,J,3),J=18,X),I=1,Y)
      WRITE(6,66)
      WRITE(6,890) ((RAY(I,J,2),J=1,17),I=1,Y)
      WRITE(6,892)
      WRITE(6,893) ((RAY(I,J,2),J=18,X),I=1,Y)
      WRITE(6,67)
      WRITE(6,890) ((RAY(I,J,1),J=1,17),I=1,Y)
      WRITE(6,892)
      WRITE(6,893) ((RAY(I,J,1),J=18,X),I=1,Y)
 65   FORMAT(/,1X,'RAY(I,J,3):')
 66   FORMAT(/,1X,'RAY(I,J,2):')
 67   FORMAT(/,1X,'RAY(I,J,1):')
      WRITE(6,68)
 68   FORMAT(/,1X,'BOUNDARY CONDITIONS:')
C ***** **** ***
      DI2=2.*DI-1.
C    IBW=BANDWIDTH
      IBW=2*X+1
      KNT=X*Y
C     DO 63 K=1,KNT
C     DO 61 L=1,IBW
C     RMAT(K,L)=0
C     RVCT(K,1)=0
 61   CONTINUE
 63   CONTINUE
      DO 64 K=1,4
      IFLAG(K)=0.
 64   CONTINUE
C
C
C LOOP 120,121 TO FORM COEFFS OF EQUATIONS FOR EACH NODE
C    FORM MATRIX RMAT TO CONTAIN DIAGONAL ELTS ONLY
      KOUNT=0
      DO 221 I=1,Y
      DO 220 J=1,X
      KOUNT=KOUNT+1
      INDEX=RAY(I,J,3)
      IF(I.EQ.1) GO TO 11
      IND1=RAY((I-1),J,3)
      TK1=2./(1./THK(IND1)+1./THK(INDEX))
 11   CONTINUE
      IF(J.EQ.1) GO TO 12
      IND2=RAY(I,(J-1),3)
      TK2=2./(1./THK(IND2)+1./THK(INDEX))
 12   CONTINUE
      IF(I.EQ.Y) GO TO 13
      IND3=RAY((I+1),J,3)
      TK3=2./(1./THK(IND3)+1./THK(INDEX))
 13   CONTINUE
      IF(J.EQ.X) GO TO 14
      IND4=RAY(I,(J+1),3)
      TK4=2./(1./THK(IND4)+1./THK(INDEX))
 14   CONTINUE
      KKK=RAY(I,J,2)
      X1=X+1
      X2=X+2
      XN=(2*X)+1
      GO TO (101,102,103,104,105,106,107,108,109,110,111,
     & 112,113,114,115,116,117,118,119,120,121,122,123,124,
     & 125,126,127,128,129,130,131,132,133),KKK
C INTERIOR NODES
 101  CONTINUE
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,X1)=-1.*(TK1+TK2+TK3+TK4)
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,XN)=TK3
      RVCT(KOUNT,1)=0.
      GO TO 200
C   CONSTANT TEMP BOUNDARYS
 102  CONTINUE
      RMAT(KOUNT,X1)=1.
      RVCT(KOUNT,1)=RAY(I,J,1)
C       SET IFLAG FOR CONST BOUND
      IF (I.NE.1) GO TO 202
      IFLAG(1)=2
      GO TO 200
 202  CONTINUE
      IF(I.NE.Y) GO TO 203
      IFLAG(3)=2
      GO TO 200
 203  CONTINUE
      IF(J.NE.1) GO TO 204
      IFLAG(2)=2
      GO TO 200
 204  CONTINUE
      IFLAG(4)=2
      GO TO 200
C   CONVECTIVE SURFACE
C                 TOP
 104  CONTINUE
      IF(I.NE.1) GO TO 143
      IFLAG(1)=4
      CONS=2.*H(INDEX)*DS
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,X1)=-1.*(TK2+2.*TK3+TK4+CONS)
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,XN)=2.*TK3
      RVCT(KOUNT,1)=-1*CONS*RAY(I,J,1)
      GO TO 200
 143  CONTINUE
C                 LEFT SIDE
      IF(J.NE.1) GO TO 149
      IFLAG(2)=4
      CONS=2.*DS*H(INDEX)
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X1)=-1.*(TK1+TK3+2.*TK4+CONS)
      RMAT(KOUNT,X2)=2.*TK4
      RMAT(KOUNT,XN)=TK3
      RVCT(KOUNT,1)=-1.*CONS*RAY(I,J,1)
      GO TO 200
 149  CONTINUE
C              BOTTOM
      IF(I.NE.Y) GO TO 141
      IFLAG(3)=4
      CONS=2.*DS*H(INDEX)
      RMAT(KOUNT,1)=2.*TK1
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,X1)=-1.*(2.*TK1+TK2+TK4+CONS)
      RVCT(KOUNT,1)=-1.*CONS*RAY(I,J,1)
      GO TO 200
 141  CONTINUE
C                RIGHT SIDE
      IFLAG(4)=4
      CONS=2.*DS*H(INDEX)
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X)=2.*TK2
      RMAT(KOUNT,X1)=-1.*(TK1+TK3+2.*TK2+CONS)
      RMAT(KOUNT,XN)=TK3
      RVCT(KOUNT,1)=-1.*CONS*RAY(I,J,1)
      GO TO 200
C   CONSTANT INTERIOR NODES
 105  CONTINUE
      WRITE(6,151) I,J,RAY(I,J,1)
 151  FORMAT(1X,'CONSTANT INTERIOR NODE RAY(',I2,',',I2,',','1)',
     & '=',F6.2)
      RMAT(KOUNT,X1)=1.
      RVCT(KOUNT,1)=RAY(I,J,1)
      GO TO 200
C  CONSTANT HEAT FLUX BOUNDARY
 103  CONTINUE
C         TOP
      IF(I.NE.1) GO TO 161
      IFLAG(1)=6
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,X1)=-1.*((TK2)+(2.*TK3)+(TK4))
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,XN)=2.*TK3
      RVCT(KOUNT,1)=-FI(1)*DS
      GO TO 200
 161  CONTINUE
C         BOTTOM
      IF(I.NE.Y) GO TO 162
      IFLAG(3)=6
      RMAT(KOUNT,1)=2.*TK1
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,X1)=-1.*((2.*TK1)+(TK2)+(TK4))
      RMAT(KOUNT,X2)=TK4
      RVCT(KOUNT,1)=-FI(3)*DS
      GO TO 200
 162  CONTINUE
C         LEFT SIDE
      IF(J.NE.1) GO TO 163
      IFLAG(2)=6
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X1)=-1.*((TK1)+(TK3)+(2.*TK4))
      RMAT(KOUNT,X2)=2.*TK4
      RMAT(KOUNT,XN)=TK3
      RVCT(KOUNT,1)=-FI(2)*DS
      GO TO 200
 163  CONTINUE
C         RIGHT SIDE
      IFLAG(4)=6
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X)=2.*TK
      RMAT(KOUNT,X1)=-1.*((TK1)+(2.*TK2)+(TK3))
      RMAT(KOUNT,XN)=TK3
      RVCT(KOUNT,1)=-FI(4)*DS
      GO TO 200
 104  CONTINUE
C          SEMI-INFINITE BOUNDARY
      IF(I.NE.1) GO TO 1060
C         TOP
      WRITE(1,1000) I,J
      GO TO [illegible]
 1060 CONTINUE
      IF(J.NE.1) GO TO 1061
C         LEFT
      IFLAG(2)=6
      DI2=2.*DI-1.
      TKL=THK(INDEX)
      RMAT(KOUNT,1)=DI2*TK1
      RMAT(KOUNT,X1)=-1.*(TK4/DI+DI2*TK1+DI2*TK3
     & +TKL/DI)
      RMAT(KOUNT,X2)=TK4/DI
      RMAT(KOUNT,XN)=DI2*TK3
      RVCT(KOUNT,1)=-1.*(TKL/DI)*RAY(I,J,1)
      GO TO 200
 1061 CONTINUE
      IF(I.NE.Y) GO TO 1062
C         BOTTOM
      IFLAG(3)=6
      DI2=2.*DI-1.
      TKB=THK(INDEX)
      RMAT(KOUNT,1)=TK1/DI
      RMAT(KOUNT,X)=DI2*TK2
      RMAT(KOUNT,X1)=-1.*(TK1/DI+DI2*TK2+DI2*TK4
     & +TKB/DI)
      RMAT(KOUNT,X2)=DI2*TK4
      RVCT(KOUNT,1)=-1.*((TKB/DI)*RAY(I,J,1))
      GO TO 200
 1062 CONTINUE
C         RIGHT
C        RIGHT
      IFLAG(4)=6
      DI2=2.*DI-1.
      TKR=THK(INDEX)
      RMAT(KOUNT,1)=DI2*TK1
      RMAT(KOUNT,X)=TK2/DI
      RMAT(KOUNT,X1)=-1.*(TK2/DI+DI2*TK1+DI2*TK3+TKR/DI)
      RMAT(KOUNT,XN)=DI2*TK3
      RVCT(KOUNT,1)=-1.*(TKR/DI)*RAY(I,J,1)
      GO TO 200
 1000 FORMAT(1X,'RAY(',I2,',',I2,') HAS NO EQUATION')
  107 CONTINUE
C        NODE ADJACENT TO LEFT SIDE SEMI-INF BOUNDARY
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X)=TK2/DI
      RMAT(KOUNT,X1)=-1.*(TK1+TK2/DI+TK3+TK4)
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,XN)=TK3
      RVCT(KUNT,1)=0.
      GO TO 200
  108 CONTINUE
C         NODE ADJACENT TO BOTTOM SEMI-INF BNDRY
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,X1)=-1.*(TK1+TK2+TK3/DI+TK4)
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,XN)=TK3/DI
      RVCT(KOUNT,1)=0.
      GO TO 200
  109 CONTINUE
C         NODE ADJACENT TO RIGHT SEMI-INFINITE BONDARY
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,X1)=-1.*(TK1+TK2+TK3+TK4/DI)
      RMAT(KOUNT,X2)=TK4/DI
      RMAT(KOUNT,XN)=TK3
      RVCT(KOUNT,1)=0.
      GO TO 200
  110 CONTINUE
C         NODE ADJACENT TO TOP SEMI-INFINITE BOUNDARY
      RMAT(KOUNT,1)=TK1/DI
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,X1)=-1.*(TK1/DI+TK2+TK3+TK4)
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,XN)=TK3
      RVCT(KOUNT,1)=0.
      GO TO 200
  111 CONTINUE
C          CONSTANT CORNER
      RMAT(KOUNT,X1)=1.
      RVCT(KOUNT,1)=RAY(I,J,1)
      WRITE(6,171) I,J,RAY(I,J,1)
  171 FORMAT(1X,'CONSTANT CORNER RAY(',I2,',',I2,',','1)=',
     *F8.2)
      GO TO 200
  112 CONTINUE
      IF(J.NE.1) GO TO 1120
      IF(I.NE.1) GO TO 1121
C          TOP LEFT CORNER CONST FLUX
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,XN)=TK3
      RMAT(KOUNT,X1) =-1.*(TK3+TK4)
      RVCT(KOUNT,1)=-.5*(FI(1)+FI(2))
      WRITE(6,207)
  207 FORMAT(1X,'TOP LEFT CORNER CONST FLUX')
      GO TO 200
 1121 CONTINUE
C          BOTTOM LEFT CORNER CONST FLUX
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X1)=-1.*(TK4+TK1)
      RVCT(KOUNT,1)=-.5*(FI(2)+FI(3))
      WRITE(6,208)
  208 FORMAT(1X,'BOTTOM LEFT CORNER CONST FLUX')
      GO TO 200
 1120 CONTINUE
      IF(I.NE.Y) GO TO 1122
C          BOTTOM RIGHT CORNER CONST FLUX
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,X1)=-1.*(TK1+TK2)
      RVCT(KOUNT,1)=-.5*(FI(3)+FI(4))
      WRITE(6,209)
  209 FORMAT(1X,'BOTTOM RIGHT CORNER CONST FLUX')
      GO TO 200
 1122 CONTINUE
C          TOP RIGHT CORNER CONST FLUX
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,XN)=TK3
      RMAT(KOUNT,X1)=-1.*(TK2+TK3)
      RVCT(KOUNT,1)=-.5*(FI(1)+FI(4))
      WRITE(6,210)
 210   FORMAT(1X,'TOP RIGHT CORNER CONST FLUX')
       GO TO 200
 113   CONTINUE
       IF(J.NE.1) GO TO 1130
       IF(I.NE.1) GO TO 1131
C            TOP LEFT CORNER CONVECTIVE
       CONS=2.*DS*H(INDEX)
       RMAT(KOUNT,X1)=-1.*(TK3+TK4+CONS)
       RMAT(KOUNT,XN)=TK3
       RMAT(KOUNT,X2)=TK4
       RVCT(KOUNT,1)=-1.*CONS*RAY(I,J,1)
       WRITE(6,211)
 211   FORMAT(1X,'TOP LEFT CORNER CONVECTIVE')
       GO TO 200
 1131   CONTINUE
C            BOTTOM LEFT CORNER CONVECTIVE
       CONS=2.*DS*H(INDEX)
       RMAT(KOUNT,1)=TK1
       RMAT(KOUNT,X1)=-1.*(TK1+TK4+CONS)
       RMAT(KOUNT,X2)=TK4
       RVCT(KOUNT,1)=-1.*CONS*RAY(I,J,1)
       WRITE(6,212)
 212   FORMAT(1X,'BOTTOM LEFT CORNER CONVECTIVE')
       GO TO 200
 1130  CONTINUE
       IF(I.NE.Y) GO TO 1132
C            BOTTOM RIGHT CORNER CONVECTIVE
       CONS=2.*DS*H(INDEX)
       RMAT(KCUNT,1)=TK1
       RMAT(KOUNT,X)=TK2
       RMAT(KOUNT,X1)=-1.*(TK1+TK2+CONS)
       RVCT(KOUNT,1)=-1.*CONS*RAY(I,J,1)
       WRITE(6,213)
 213   FORMAT(1X,'BOTTOM RIGHT CORNER CONVECTIVE')
       GO TO 200
 1132  CONTINUE
C            TOP RIGHT CORNER CONVECTIVE
       CONS=2.*DS*H(INDEX)
       RMAT(KOUNT,X)=TK2
       RMAT(KOUNT,X1)=-1.*(TK2+TK3+CONS)
       RMAT(KOUNT,XN)=TK3
       RVCT(KOUNT,1)=-1.*CONS*RAY(I,J,1)
       WRITE(6,214)
 214   FORMAT(1X,'TOP RIGHT CORNER CONVECTIVE')
       GO TO 200
 116   CONTINUE
C                  CORNER: VERT=CONVECT,HORIZ=CONST FLUX
       IF(I.EQ.1) PHI=FI(1)
       IF(I.EQ.Y) PHI=FI(3)
       GO TO 1160
 117   CONTINUE
C                CORNER: HORIZ=CONVECT,VERT=CONST FLUX
       IF(J.EQ.1) PHI=FI(2)
       IF(J.EQ.X) PHI=FI(4)
 1160  CONTINUE
       IF(J.NE.1) GO TO 1161
       IF(I.NE.1) GO TO 1162
C           TOP LEFT CORNER CONVECT, CFLUX
       CONS=DS*H(INDEX)
       RMAT(KOUNT,XN)=TK3
       RMAT(KOUNT,X2)=TK4
       RMAT(KOUNT,X1)=-1.*(TK3+TK4+CONS)
       RVCT(KOUNT,1)=-1.*CONS*RAY(I,J,1)-PHI*DS
       WRITE(6,215)
 215   FORMAT(1X,'TOP LEFT CORNER CONVECTIVE & C FLUX')
       GO TO 200
 1162  CONTINUE
C         BOTTOM LEFT CORNER CONVECT, C FLUX
       CONS=DS*H(INDEX)
       RMAT(KOUNT,1)=TK1
       RMAT(KOUNT,X2)=TK4
       RMAT(KOUNT,X1)=-1.*(TK1+TK4+CONS)
       RVCT(KOUNT,1)=-1.*CONS*RAY(I,J,1)-PHI*DS
       WRITE(6,216)
 216   FORMAT(1X,'BOTTOM LEFT CORNER CONVECTIVE & CONST FLUX')
       GO TO 200
 1161  CONTINUE
       IF(I.NE.Y) GO TO 1163
C                BOTTOM RIGHT CORNER CONVECT, C FLUX
       CONS=DS*H(INDEX)
       RMAT(KOUNT,1)=TK1
       RMAT(KOUNT,X)=TK2
       RMAT(KOUNT,X1)=-1.*(TK1+TK2+CONS)
       RVCT(KOUNT,1)=-1.*CONS*RAY(I,J,1)
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      WRITE(6,217)
 217  FORMAT(1X,'BOTTOM RIGHT CORNER CONVECT & C FLUX')
      GO TO 200
 1163 CONTINUE
C          TOP RIGHT CORNER CONVECT,C FLUX
      CONS=DS*H(INDEX)
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,XN)=TK3
      RMAT(KOUNT,X1)=-1.*(TK2+TK3+CONS)
      RVCT(KOUNT,1)=-1.*CONS*RAY(I,J,1)-PHI*DS
      WRITE(6,218)
 218  FORMAT(1X,'TOP RIGHT CORNER CONVECT, CONST FLUX')
      GO TO 200
 114  CONTINUE
      IF(J.NE.1) GO TO 1140
      IF(I.NE.1) GO TO 1141
C                 TOP LEFT CORNER SEMI-INFINITE ON BOTH SIDES
      WRITE(1,1000) I,J
      GO TO 298
 1141 CONTINUE
C                  BOTTOM LEFT CORNER SEMI-INFINITE ON BOTH SIDES
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X2)=TK4
      TKI=TK1
      TKL=TK1
      TKB=TK1
      TLFT=RAY(I,J,1)
      TBTM=RAY(I,J,1)
      DI=100.
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,X1)=-1.*(TK1+TK4+TKL+TKB)
      RVCT(KOUNT,1)=-TKL*TLFT-TKB*TBTM
      WRITE (6,221)
 221  FORMAT(1X,'BOTTOM LEFT CORNER SEMI-INFINITE')
      GO TO 200
 1140 CONTINUE
      IF(I.NE.Y) GO TO 1142
C               BOTTOM RIGHT CORNER SEMI-INFINITE ON BOTH SIDES
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X)=TK2
      TKB=TKI
      TKR=TK1
      TRIT=RAY(I,J,1)
      TBTM=RAY(I,J,1)
      DI=100.
      RMAT(KOUNT,X1)=-1.*(TK1+TK2+TKR+TKB)
      RVCT(KOUNT,1)=-TKR*TRIT-TKB*TBTM
      WRITE(6,222)
 222  FORMAT(1X,'BOTTOM RIGHT CORNER SEMI-INFINITE')
      GO TO 200
 1142 CONTINUE
C           TOP RIGHT CORNER SEMI-INFINITE ON BOTH SIDES
      WRITE(1,1000) I,J
      GO TO 298
 115  CONTINUE
C      SQUARE INTERIOR NODE NEXT TO TWO SEMI-INF SIDES
      IF(J.NE.2) GO TO 1150
      IF(I.NE.2) GO TO 1151
C          TOP LEFT
      WRITE(1,1000) I,J
      GO TO 298
 1151 CONTINUE
C          BOTTOM LEFT
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X)=TK2/DI
      RMAT(KOUNT,X1)=-1.*(TK1+TK2/DI+TK3/DI+TK4)
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,XN)=TK3/DI
      RVCT(KOUNT,1)=0.
C     WRITE(6,226) I,J
 226  FORMAT(1X,'RAY(',I2,',',I2,'2)=15')
      GO TO 200
 1150 CONTINUE
      YY=Y-1
      IF(I.NE.YY) GO TO 1152
C             BOTTOM RIGHT
      RMAT(KOUNT,1)=TK1
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,X1)=-1.*(TK1+TK2+TK3/DI+TK4/DI)
      RMAT(KOUNT,X2)=TK4/DI
      RMAT(KOUNT,XN)=TK3/DI
      RVCT(KOUNT,1)=0.
      GO TO 200
 1152 CONTINUE
C            TOP RIGHT
      WRITE(1,1000) I,J
      GO TO 998
 118  CONTINUE
C       BOTTOM RIGHT CORNER HORIZ SEMI-INNF,VERT CONST FLX
      DI2=2.*DI-1.
      TKB=TK1
      RMAT(KOUNT,1)=TK1/(DI*2.)
      RMAT(KOUNT,X)=TK2*DI2
      RMAT(KOUNT,X1)=-1.*(TK1/(DI*2.)+TK2*DI2+TKB/(DI*2.))
      RVCT(KOUNT,1)=-1.*(TKB/(DI*2.))*RAY(I,J,1)-FI(4)*DI2*DS
      GO TO 200
 119  CONTINUE
C     BOTTOM RIGHT SQUARE NEXT TO SEMI-INF,RT SIDE CONST FLX
      DI2=2.*DI-1.
      RMAT(KOUNT,1)=TK1/2.
      RMAT(KOUNT,X)=TK2
      RMAT(KOUNT,X1)=-1.*(TK1/2.+TK2+TK3/(DI*2.))
      RMAT(KOUNT,XN)=TK3/(DI*2.)
      RVCT(KOUNT,1)=-FI(4)*DS
      GO TO 200
 123  CONTINUE
C     BOTTOM LEFT SQUARE NET TO SEMI-INF,LEFT SIDE CONST FLX
      DI2=2.*DI-1.
      RMAT(KOUNT,1)=TK1/2.
      RMAT(KOUNT,X1)=-1.*(TK1/2.+TK4+TK3/(DI*2.))
      RMAT(KOUNT,X2)=TK4
      RMAT(KOUNT,XN)=TK3/(DI*2.)
      RVCT(KOUNT,1)=-FI(2)*DS
      GO TO 200
 122  CONTINUE
C     BOTTOM LEFT CORNER HORIZ SEMI-INF,VERT CONST FLX
      DI2=2.*DI-1.
      TKB=TK1
      RMAT(KOUNT,1)=TK1/(DI*2.)
      RMAT(KOUNT,X1)=-1.*(TK1/(DI*2.)+TK4*DI2+TK3/(DI*2.))
      RMAT(KOUNT,X2)=TK4*DI2
      RVCT(KOUNT,1)=-1.*(TKB/(DI*2.))*RAY(I,J,1)
      GO TO 200
 130  CONTINUE
C        SEMI-INF NODE ABOVE SEMI-INF CORNER NODE
      IF(J.NE.1) GO TO 1300
C                  LEFT SIDE
      TK3=DI/(((DI-.5)/THK(IND3))+(.5/THK(INDEX)))
      TK4=DI/(((DI-.5)/THK(INDEX))+(.5/THK(IND4)))
      RMAT(KOUNT,1)=TK1*DI2
      RMAT(KOUNT,X2)=TK4/DI
      RMAT(KOUNT,XN)=TK3*(DI2/DI)
      RMAT(KOUNT,X1)=-1.*(TK1*DI2+TK4/DI+TK3*DI2/DI+THK(INDEX)/DI)
      RVCT(KOUNT,1)=-1.*(THK(INDEX)/DI)*RAY(I,J,1)
      GO TO 200
 1300 CONTINUE
C                    RIGHT SIDE
      TK2=DI/(((DI-.5)/THK(INDEX))+(.5/THK(IND2)))
      TK3=DI/(((DI-.5)/THK(IND3))+(.5/THK(INDEX)))
      RMAT(KOUNT,1)=TK1*DI2
      RMAT(KOUNT,X)=TK2/DI
      RMAT(KOUNT,XN)=TK3*(DI2/DI)
      RMAT(KOUNT,X1)=-1.*(TK1*DI2+TK2/DI+TK3*DI2/DI+THK(INDEX)/DI)
      RVCT (KOUNT,1)=-1.*(THK(INDEX)/DI)*RAY(I,J,1)
      GO TO 200
 133  CONTINUE
C         SEMI-INF NODE TO RIGHT OF SEMI-INF CORNER NODE
      IF(I.NE.1) GO TO 1330
C                TOP
      WRITE(1,1000) I,J
      GO TO 998
 1330 CONTINUE
C              BOTTOM
      TK1=DI/(((DI-.5)/THK(INDEX))+(.5/THK(IND1)))
      TK2=DI/(((DI-.5)/THK(IND2))+(.5/THK(INDEX)))
      RMAT(KOUNT,1)=TK1/DI
      RMAT(KOUNT,X)=TK2*DI2/DI
      RMAT(KOUNT,X1)=-1.*(TK1/DI+TK2*DI2/DI+TK4*DI2+THK(INDEX)/DI)
      RMAT(KOUNT,X2)=TK4*DI2
      RVCT(KOUNT,1)=-1.*(THK(INDEX)/DI)*RAY(I,J,1)
      GO TO 200
 120  CONTINUE
 121  CONTINUE
 124  CONTINUE
 125  CONTINUE
 126  CONTINUE
 127  CONTINUE
 129  CONTINUE
 129  CONTINUE
 132  CONTINUE
 131  CONTINUE
      WRITE(1,1000) I,J
      GO TO 998
 200  CONTINUE
 201  CONTINUE
C ---- ---- ----- -----
C  LOOP TO WRITE BOUNDARY TYPES ONTO STDOUT
      DO 90 K=1,4
      IF (K.NE.1) GO TO 9001
      WRITE(6,9005)
 9005 FORMAT(1X,'TOP BOUNDARY')
      GO TO 9009
 9001 CONTINUE
      IF (K.NE.2) GO TO 9002
      WRITE(6,9006)
 9006 FORMAT(1X,'LEFT BOUNDARY')
      GO TO 9009
 9002 CONTINUE
      IF (K.NE.3) GO TO 9003
      WRITE(6,9007)
 9007 FORMAT(1X,'BOTTOM BOUNDARY')
      GO TO 9009
 9003 CONTINUE
      WRITE(6,9008)
 9008 FORMAT(1X,'RIGHT BOUNDARY')
 9009 CONTINUE
      IF(IFLAG(K).NE.1) GO TO 91
      WRITE(6,95)
      GO TO 90
 91   CONTINUE
      IF (IFLAG(K).NE.4) GO TO 93
      WRITE(6,97)
      GO TO 90
 93   CONTINUE
      IF (IFLAG(K).NE.5) GO TO 94
      WRITE(6,98)
      GO TO 90
 94   CONTINUE
      IF(IFLAG(K).NE.3) GO TO 92
      WRITE(6,96)
      GO TO 90
 92   CONTINUE
 95   FORMAT(1H+,'                         CONSTANT TEMPERATURE')
 97   FORMAT(1H+,'                         CONVECTIVE SURFACE')
 98   FORMAT(1H+,'                         CONSTANT HEAT FLUX')
 96   FORMAT(1H+,'                         SEMI-INFINITE')
 90   CONTINUE
C ----      ----    ----   ----   ----
      WRITE(6,72) IB
      WRITE(1,72) IB
 72   FORMAT(/,1X,'BANDWIDTH=',I5)
C SOLVE MATRIX RMAT
      NLC=X
      NUC=X
      N=Y*X
      IA=Y*X
      M=1
      IB=Y*X
      IJOB=0
      IER=678
      WRITE(1,80)
      WRITE(6,80)
 80   FORMAT(1X,' NLC, NUC,   N,  IA,    M,   IB,IJOB, IER:')
      WRITE(1,81) NLC,NUC,N,IA,M,IB,IJOB,IER
      WRITE(6,81) NLC,NUC,N,IA,M,IB,IJOB,IER
 81   FORMAT(1X,8I5)

      CALL BANDMX(N,NLC,NUC,IA,M,IB,IJOB,IER)
      WRITE(1,74) IER
      WRITE(6,74) IER
 74   FORMAT(/,1X,'IER=',I3,':',/,5X,'IF IER=129 , MATRIX IS SINGULAR',
     & /,5X,'IF IER=0, EVERYTHING IS OKAY.')
      IF(IER.EQ.129) GO TO 998
      K=0
      DO 310 I=1,Y
      DO 300 J=1,X
      K=K+1
      RAY(I,J,1)=RVCT(K,1)
 300  CONTINUE
 310  CONTINUE
      WRITE(6,891)
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891   FORMAT(1X,'FINAL TEMPERATURE DISTRIBUTION:')
      WRITE(6,890) ((RAY(I,J,1),J=1,17),I=1,Y)
      WRITE(6,892)
      WRITE(6,893) ((RAY(I,J,1),J=18,X),I=1,Y)
890   FORMAT(1X,17F7.2)
892   FORMAT(/,1X,'..THE REST OF THE COLUMNS:')
893   FORMAT(1X,4F7.2)
      CALL ISOTHM
      GO TO 998
999   CONTINUE
      WRITE(1,9999)
9999  FORMAT(1X,'ERROR IN ASSIGNING NODAL LOCATIONS')
998   CONTINUE
      CALL CONTRL(4,'STSDAT',5)
      CALL CONTRL(4,'STDOUT',6)
      CALL EXIT
      END
C
C ISOTHM FINDS ISOTHERMS FOR SSCONDUCT-------------------
      SUBROUTINE ISOTHM
      IMPLICIT INTEGER(A,B,D,W,X,Y,Z)
      COMMON/M1I/ A,X,Y,NISO
      COMMON/M1R/ DS,DI,TMAX,TMIN,FI(4),THK(2),H(2),TISO(9),
     & RAY(100,100,3)
      DIMENSION EXRY(9,60), EYRY(9,60), COUNT(9)
      DIMENSION IX(5),IY(3)
      CALL CONTRL(2,'POINTS',7)
      CNTPT=0
      DO 5 L=1,9
      COUNT(L)=0
 5    CONTINUE
C   CHANGE THE NXT TWO STMTS TO AGREE WITH DIMENSION:
      DO 6 K=1,40
      DO 7 B=1,9
      EXRY(B,K)=0
      EYRY(B,K)=0
 7    CONTINUE
 6    CONTINUE
C   TMAX=TEMP OF HOTTEST ISOTHERM
C   TMIN=TEMP OF COLDEST ISOTHERM
C   NISO= NO OF ISOTHERMS
C         LET 1= HOTTEST ISOTHERM
C   COUNT(B)=COUNTER FOR NO. ELTS IN EACH ISOTHERM
C   TISO(B)= TEMP OF EACH ISOTHERM
C   EXRY(B,K)=ARRAY FOR X-COORDINATES
C         B INDICATES WHICH ISOTHERM
C         K=COUNT(B) & INDICATES POSITION OF PT IN ISOTHERM LIST
C   LOOP TO LOCATE ISOTHERMS
C          FOR LEFT SEMI-INF,LET XL=3
C          FOR BOTTOM SEMI-INF,LET YB=Y-3
C          FOR RIGHT SEMI-INF,LET XR=X-3, AND COMMENT OUT LOOP 800
C          FOR TOP SEMI-INF,LET YT=3, AND COMMENT OUT LOOP 860
C          IF NOT USING SEMI-INF,XL=1,XR=X-1,YT=2,YB=Y
      XL=3
      XR=X-1
      YB=Y-3
      YT=2
      DO 100 I=YT,YB
      DO 200 J=XL,XR
C
C EXAMINE TEMPS HORIZONTALLY
      RJ=RAY(I,J,1)
      RJ1=RAY(I,(J+1),1)
C     IF((RJ.GT.TISO(1)).AND.(RJ1.GT.TISO(1))) GO TO 500
C     IF((RJ.LT.TISO(NISO)).AND.(RJ1.LT.TISO(NISO))) GO TO 500
      DO 500 B=1,NISO
      IF((TISO(B).GT.RJ).AND.(TISO(B).LT.RJ1)) GO TO 400
      IF((TISO(B).LT.RJ).AND.(TISO(B).GT.RJ1)) GO TO 400
      IF(TISO(B).EQ.RJ) GO TO 402
      GO TO 500
400   CONTINUE
C     WRITE(7,401) I,J,RAY(I,J,1)
401   FORMAT(1X,'RAY(',I3,I3,',1)=',F6.2)
      EXCD=(RAY(I,J,1)-TISO(B))/(RAY(I,J,1)-RAY(I,(J+1),1))+J
      EXCO=DS*(EXCD-1)
      COUNT(B)=COUNT(B)+1
      K=COUNT(B)
      EXRY(B,K)=EXCO
      EYRY(B,K)=DS*(I-1)
      CNTPT=CNTPT+1
      GO TO 500
402   CONTINUE
      COUNT(B)=COUNT(B)+1
      K=COUNT(B)
      EXRY(B,K)=DS*(J-1)
      EYRY(B,K)=DS*(I-1)
      CNTPT=CNTPT+1
  500 CONTINUE
C EXAMINE TEMPS VERTICALLY
      RI=RAY(I,J,1)
      RI1=RAY((I-1),J,1)
      IF((RI.GT.TISO(1)).AND.(RI1.GT.TISO(1))) GO TO 525
      IF((RI.LT.TISO(NISO)).AND.(RI1.LT.TISO(NISO))) GO TO 525
      DO 525 B=1,NISO
      IF((TISO(B).GT.RI).AND.(TISO(B).LT.RI1)) GO TO 550
      IF((TISO(B).LT.RI).AND.(TISO(B).GT.RI1)) GO TO 550
      GO TO 525
  550 CONTINUE
C     WRITE(7,401) I,J,RAY(I,J,1)
      EYCO=(RAY((I-1),J,1)-TISO(B))/(RAY((I-1),J,1)-RAY(I,J,1))+(I-1)
      EYCO=DS*(EYCO-1)
      COUNT(B)=COUNT(B)+1
      K=COUNT(B)
      EYRY(B,K)=EYCO
      EXRY(B,K)=DS*(J-1)
      CNTPT=CNTPT+1
  525 CONTINUE
  200 CONTINUE
  100 CONTINUE
C   800 LOOP IS FOR RIGHT HAND SIDE
      DO 800 I=YT,YB
      DO 851 B=1,NISO
      RI=RAY(I,X,1)
      RI1=RAY((I-1),X,1)
      IF((RI.GT.(TISO(B))).AND.(RI1.LT.(TISO(B)))) GO TO 852
      IF((RI.LT.(TISO(B))).AND.(RI1.GT.(TISO(B)))) GO TO 852
      IF(RI.EQ.TISO(B)) GO TO 853
      GO TO 851
  852 CONTINUE
      EYCO=(I-1)+(RI1-TISO(B))/(RI1-RAY(I,X,1))
      EYCO=DS*(EYCO-1)
      COUNT(B)=COUNT(B)+1
      K=COUNT(B)
      EYRY(B,K)=EYCO
      EXRY(B,K)=DS*(X-1)
      CNTPT=CNTPT+1
      GO TO 851
  853 CONTINUE
      COUNT(B)=COUNT( )+1
      K=COUNT(B)
      EYRY(B,K)=DS*(I-1)
      EXRY(B,K)=DS*(X-1)
      CNTPT=CNTPT+1
  851 CONTINUE
  800 CONTINUE
C             860 LOOP IS FOR TOP ROW
      XX=X-1
      I=1
      DO 860 J=XL,XR
      RJ=RAY(1,J,1)
      RJ1=RAY(1,(J+1),1)
      DO 861 B=1,NISO
      IF((TISO(B).GT.RJ).AND.(TISO( ).LT. J1)) GO TO 862
      IF((TISO(B).LT.RJ).AND.(TISO( ).GT.RJ1)) GO TO 862
      IF(RJ.EQ.TISO(B)) GO TO 863
      GO TO 861
  862 CONTINUE
      EXCO=(RAY(I,J,1)-TISO(B))/(RAY(I,J,1)-RAY(I,(J+1),1))+J
      EXCO=DS*(EXCO-1)
      COUNT(B)=COUNT( )+1
      K=COUNT(B)
      EXRY(B,K)=EXCO
      EYRY(B,K)=0
      CNTPT=CNTPT+1
      GO TO 861
  863 CONTINUE
      COUNT(B)=COUNT( )+1
      K=COUNT(B)
      EXRY(B,K)=DS*(J-1)
      EYRY(B,K)=0
      CNTPT=CNTPT+1
  861 CONTINUE
  860 CONTINUE
      WRITE(7,92) CNTPT
      DO 666 B=1,NISO
      L=COUNT(B)
      WRITE(1,91) B,L
      WRITE(7,91) B,L
```

```
 91    FORMAT(1X,'ISOTHM #',I2,' HAS',I4,' POINTS')
       IF(L.EQ.0) GO TO 666
       WRITE(7,90) (EXRY(B,K),EYRY(B,K),K=1,L)
 90    FORMAT(1X,2F15.3)
 666   CONTINUE
       WRITE(1,93) CNTPT
 93    FORMAT(1X,'TOTAL NO. POINTS FOUND=',I5)
 92    FORMAT(1X,I5)
       CALL CONTRL(4,'POINTS',7)
       RETURN
       END
C ***************************************************
C SSDATA, DATA FOR SSCONDUCT
       SUBROUTINE SSDATA
       IMPLICIT INTEGER(A,B,Q,W,X,Y,Z)
       COMMON/M1I/ A,X,Y,NISO
       COMMON/M1R/ DS,DI,TMAX,TMIN,FI(4),THK(2),H(2),TISO(9),
      $ RAY(100,100,3)
C     ALL UNITS ARE IN METERS,HOURS,CELSIUS, BUT
C     IF YOU USE ENGLISH UNITS,MAKE SURE THAT
C       THEY ARE CONSISTENT.
C    DS=DISTANCE BETWEEN NODES (M)
C    TSRF=SURFACE TEMP (OUTSIDE GRID) (INFINITE DIST AWAY)
C    TBTM=BOTTOM TEMP (UNDER GRID) (INFINITE DIST AWAY)
C    TRIT=TEMP TO RIGHT OF GRID (INFINITE DIST)
C    TLFT=TEMP TO LEFT OF GRID  (INFINITE DIST AWAY)
C    X= NO. OF GRID NODES HORIZONTALLY
C    Y= NO. OF GRID NODES VERTICALLLY
C    A= NO. OF DIFFERENT MATERIALS IN THE GRID
C    DI= DISTANCE HALFWAY TO INFINITY (DI IS THE MULTIPLE OF DS)
C    THK(L)=THERMAL CONDUCTIVITY (W/M*K)
C    H(L)=CONVECTION COEFFICIENT
C    FI(1)= HEAT FLUX FROM TOP OF GRID
C    FI(2)= HEAT FLUX FROM LEFT SIDE OF GRID
C    FI(3)= HEAT FLUX FROM BOTTOM OF GRID
C    FI(4)= HEAT FLUX FROM RIGHT SIDE
C    RAY(I,J,1)=PRESENT NODAL TEMP
C    RAY(I,J,2)=LOCATION TYPE:
C                    =1 INTERIOR NODE, FLUCTUATING TEMP
C                    =2 NODE ON CONSTANT TEMP BOUNDARY
C                    =3 ON CONSTANT FLUX BOUNDARY
C                    =4 ON CONVECTIVE SURFACE BOUNDARY
C                    =5 INTERIOR NODE, CONSTANT TEMP
C                    =6 SEMI-INFINITE BOUNDARY
C                    =7    NODE ADJACENT TO LEFT SEMI-INF BNDRY
C                    =8    NODE ADJACENT TO BOTTOM SEMI-INF BNDRY
C                    =9    NODE ADJACENT TO RIGHT SEMI-INF BNDRY
C                    =10   NODE ADJACENT TO TOP SEMI-INF BNDRY
C                    =11 CONSTANT CORNER NODE
C                    =12 CONST FLUX (ON BOTHE SIDES) CORNER NODE
C                    =13 CONVECTIVE(ON BOTH SIDES) CORNER NODE
C                    =14 SEMI-INF (ON BOTH SIDES) CORNER NODE
C                    =15    SQUARE NODE ADJACENT TO TWO SEMI-INF SIDES
C                    =30    SEMI-INF NODE ABOVE SEMI-INF CORNER NODE
C                    =31    SEMI-INF NODE TO LEFT OF SEMI-INF CORNER NODE
C                    =32    SEMI-INF NODE BELOW SEMI-INF CORNER NODE
C                    =33    SEMI-INF NODE TO RIGHT OF SEMI-INF CORNER NODE
C                    =16 CORNER WITH VERTICAL SIDE CONVECT,HORIZ SIDE CONST FLUX
C                    =17 CORNER WITH HORIZ SIDE CONVECT,VERT SIDE CONST FLUX
C                    =18 BOTTOM RIGHT CORNER. VERT=CONST FLUX,HORIZ=SEMI-INF
C                    =19    RIGHT SIDE CONST FLUX NODE ADJACENT TO BOTTOM SEMI-INF
C                    =20 TOP LEFT CORNER. VERT=SEMI-INF,HORIZ=CONST FLUX
C                    =21    TOP CONST FLUX NODE ADJACENT TO LEFT SEMI-INF
C                    =22 BOTTOM LEFT COENER. VERT=CONST FLUX,HORIZ=SEMI-INF
C                    =23    LEFT SIDE CONST FLUX ADJACENT TO BOTTOM SEMI-INF
C                    =24 BOTTOM RIGHT CORNER. VERT=CONVECT, HOEIZ=SEMI-INF
C                    =25    RIGHT SIDE CONVECT NODE ADJAC TO BOTTOM SEMI-INF
C                    =26 TOP LEFT CORNER. VERT=SEMI-INF,HORIZ=CONVECT
C                    =27    TOP CONVECT NODE ADJAC TO LEFT SEMI-INF
C                    =28 BOTTOM LEFT CORNER. VERT=CONVECT,HORIZ=SEMI-INF
C                    =29    LEFT SIDE CONVECT NODE ADJACENT TO BOTTOM SEMI-INF
C    RAY(I,J,3)= INDEX OF MATERIAL (RANGES FROM 1 TO A)
C    NISO= NO. OF ISOTHERMS TO BE PLOTTED
C    TMAX= TEMP OF HOTTEST ISOTHERM (*C)
C    TMIN= TEMP OF COLDEST ISOTHERM (*C)
C    TISO(L)= ARRAY CONTAINING ISOTHERM TEMPS
C            TISO(1) HAS THE HOTTEST ISOTHERM
C    COUNT(3)=COUNTER FOR NO.ELTS IN EACH ISOTHERM
C
C INITIALIZE VARIABLES
       DS=.1
       A=1
       X=21
       Y=31
      DT=50.
C     TSRF...TBTM ONLY MATTER FOR CONST OR CONVECTIVE OR SEMI-INF BOUNDARY.
C     FOR CONVECTIVE BNDRY,TSRF...TBTM INDICATES TEMP AWAY FROM GRID.
C     FOR CONSTANT BOUNDRY,TSRF...TBTM INDICATE THE TEMP OF THE BOUNDARY.
C     FOR SEMI-INF BOUNDRY,TSRF...TBTM INDICATE THE TEMP AT INFINITY.
      TSRF=10.0
      TLFT=10.0
      TRIT=10.0
      TBTM=10.0
C
C   THE FOLLOWING VALUES MATTER ONLY IF YOU INTEND TO LOCATE
C   AND PLOT THE ISOTHERMS OF THE RESULTING TEMPERATURE
C   DISTRIBUTION.   IF YOU DO NOT WISH TO RUN SUBROUTINE
C   ISOTHM, LEAVE THE FOLLOWING 7 LINES AS THEY ARE.
      TMAX=100.0
      TMIN=25
      NISO=4
C   INITIALIZE ISOTHERM TEMPS
C     IN ORDER, WITH TISO(1) HOTTEST
      TISO(1)=100.0
      TISO(2)=70.0
      TISO(3)=50.0
      TISO(4)=30.0
C
C       INITIALIZE FLUXES FROM BOUNDARIES:
C           SET TO ZERO IF NOT USED OR IF INSULATED.
      FI(1)=0.
      FI(2)=0.
      FI(3)=0.
      FI(4)=0.
C
C   DO NOT CHANGE THE FOLLOWING 11 LINES
      DO 5 J=1,X
      DO 6 I=1,Y
      DO 7 K=1,3
      RAY(I,J,K)=0
 7    CONTINUE
 6    CONTINUE
 5    CONTINUE
      DO 8 K=1,A
      THK(K)=0
      H(K)=0
 8    CONTINUE
C
C
C SET UP RAY(I,J,3) INDEX OF MATERIALS
C   THERE ARE A MATERIALS. LET THE SURROUNDING
C   MATERIAL BE THE "ATH" MATERIAL.  START
C   WITH "1".
C   INDICATE RAY(I,J,3) FOR MATERIALS #1 TO #(A-1):
C
C THIS LOOP ASSIGNS THE "ATH" MATERIAL TO
C   THE REMAINING NODES. DO NOT CHANGE THE LOOP.
      DO 10 J=1,X
      DO 20 I=1,Y
      IF (RAY(I,J,3).NE.0) GO TO 20
      RAY(I,J,3)=A
 20   CONTINUE
 10   CONTINUE
C SET UP MATERIAL PROPERTIES FOR EACH MATERIAL.
      THK(A)=1.44
      H(A)=0
C
C
C       SET UP RAY(I,J,2) NODAL LOCATION TYPE
C          BOUNDARYS
C CHANGE ONLY THE "RAY(I,J,2)=_ " STATEMENT.
      YY=Y-1
C   TOP BOUNDARY
      DO 40 J=1,X
      RAY(1,J,2)=2
      RAY(1,J,1)=TSRF
 40   CONTINUE
C   BOTTOM BOUNDARY
      DO 50 J=1,X
      RAY(Y,J,2)=6
      RAY(Y,J,1)=TBTM
 50   CONTINUE
C   LEFT BOUNDARY
      DO 60 I=2,YY
      RAY(I,1,2)=6
      RAY(I,1,1)=TLFT
 60   CONTINUE
C   RIGHT BOUNDARY
```

```
      DO 70 I=2,YY
      RAY(I,X,2)=3
      RAY(I,X,1)=TRIT
 70   CONTINUE
C           CORNERS
C      ADJUST BOTH RAY(I,J,2) AND RAY(I,J,1)
C     TOP LEFT CORNER
      RAY(1,1,2)=11
      RAY(1,1,1)=10.00
C     BOTTOM LEFT CORNER
      RAY(Y,1,2)=14
      RAY(Y,1,1)=10.0
C     BOTTOM RIGHT CORNER
      RAY(Y,X,2)=18
      RAY(Y,X,1)=10.0
C     TOP RIGHT CORNER
      RAY(1,X,2)=11
      RAY(1,X,1)=10.0
C INTERIOR NODES GIVEN FLUCTUATING TEMP
C   DON'T CHANGE THIS LOOP.
      YY=Y-1
      XX=X-1
      DO 90 J=2,XX
      DO 80 I=2,YY
      RAY(I,J,2)=1
 80   CONTINUE
 90   CONTINUE
C  INDICATE LOCATION OF ALL INTERIOR NODES OF CONST TEMP
C    SET RAY(I,J,2)=5 FOR LOCATION TYPE:
      RAY(10,X,2)=5
      RAY(11,X,2)=5
      RAY(11,(X-1),2)=5
      RAY(12,X,2)=5
C THE FOLLOWING IS FOR LEFT SIDE & BOTTOM SEMI-INFINITE.
C ADJUST THESE LOOPS AND NUMBERS TO PROVIDE FOR
C NODES ADJACENT TO SEMI-INFINITE BOUNDARY NODES.
      YY=Y-1
      DO 41 I=2,YY
      RAY(I,2,2)=7
 41   CONTINUE
      XX=X-1
      DO 42 J=2,XX
      RAY((Y-1),J,2)=9
 42   CONTINUE
C
      RAY((Y-1),1,2)=23
      RAY((Y-1),X,2)=19
      RAY((Y-1),2,2)=15
C
C SET UP RAY(I,J,1) ...NODAL TEMPERATURES
C IF CONST TEMP BOUNDARYS ARE NOT UNIFORM TEMP,
C (TSRF,TBTM,TLFT,TRIT,RESPECTIVELY),THEN INDICATE
C TEMPERATURES OF BOUNDARY NODES HERE. ALSO INDICATE
C TEMPERATURES OF CONSTANT INTERIOR NODES HERE.
      RAY(10,X,1)=100.0
      RAY(11,X,1)=100.0
      RAY(11,(X-1),1)=100.0
      RAY(12,X,1)=100.0
C             DO NOT CHANGE THE FOLLOWING 16 LINES.
      WRITE(5,131) DS,DI
 131  FORMAT(1X,2F9.4)
      WRITE(5,132) A,X,Y,NISO
 132  FORMAT(1X,4I3)
      WRITE(5,133)(THK(L),L=1,A)
      WRITE(5,133) (H(L),L=1,A)
      WRITE(5,133) (TISO(B),B=1,NISO)
 133  FORMAT(1X,F9.4)
      WRITE(5,134) (FI(I),I=1,4)
 134  FORMAT(1X,4F9.4)
      WRITE(5,151) ((RAY(I,J,3),J=1,X),I=1,Y)
      WRITE(5,151) ((RAY(I,J,2),J=1,X),I=1,Y)
      WRITE(5,151) ((RAY(I,J,1),J=1,X),I=1,Y)
 151  FORMAT(1X,11F7.2)
      RETURN
      END
C  ****************************************
C
C
C
C    BANDMX
C-----------------------------------------------------------------------
C    USAGE                   - CALL BANDMX (A,N,NLC,NUC,IA,B,M,IB,IJOB,XL,
C                                IER)
C    ARGUMENTS    A          - INPUT/OUTPUT MATRIX OF DIMENSION N BY
C                                (NUC+NLC+1). SEE PARAMETER IJOB.
C                 N          - ORDER OF MATRIX A AND THE NUMBER OF ROWS IN
```

```
C                                  B. (INPUT)
C                 NLC      - NUMBER OF LOWER CODIAGONALS IN MATRIX A.
C                              (INPUT)
C                 NUC      - NUMBER OF UPPER CODIAGONALS IN MATRIX A.
C                              (INPUT)
C                 IA       - ROW DIMENSION OF MATRIX A EXACTLY AS
C                              SPECIFIED IN THE DIMENSION STATEMENT IN THE
C                              CALLING PROGRAM. (INPUT)
C                 B        - INPUT/OUTPUT MATRIX OF DIMENSION N BY M.
C                              ON INPUT, B CONTAINS THE M RIGHT-HAND SIDES
C                              OF THE EQUATION AX = B. ON OUTPUT, THE
C                              SOLUTION MATRIX X REPLACES B. IF IJOB = 1,
C                              B IS NOT USED.
C                 M        - NUMBER OF RIGHT HAND SIDES (COLUMNS IN B).
C                              (INPUT)
C                 IB       - ROW DIMENSION OF MATRIX B EXACTLY AS
C                              SPECIFIED IN THE DIMENSION STATEMENT IN THE
C                              CALLING PROGRAM. (INPUT)
C                 IJOB     - INPUT OPTION PARAMETER. IJOB = I IMPLIES WHEN
C                              I = 0, FACTOR THE MATRIX A AND SOLVE THE
C                                EQUATION AX = B. ON INPUT, A CONTAINS THE
C                                COEFFICIENT MATRIX OF THE EQUATION AX = B,
C                                WHERE A IS ASSUMED TO BE AN N BY N BAND
C                                MATRIX. A IS STORED IN BAND STORAGE MODE
C                                AND THEREFORE HAS DIMENSION N BY
C                                (NLC+NUC+1). ON OUTPUT, A IS REPLACED
C                                BY THE U MATRIX OF THE L-U DECOMPOSITION
C                                OF A ROWWISE PERMUTATION OF MATRIX A. U
C                                IS STORED IN BAND STORAGE MODE.
C                              I = 1, FACTOR THE MATRIX A. A CONTAINS THE
C                                SAME INPUT/OUTPUT INFORMATION AS IF
C                                IJOB = 0.
C                              I = 2, SOLVE THE EQUATION AX = B. THIS
C                                OPTION IMPLIES THAT BANDMX HAS ALREADY
C                                BEEN CALLED USING IJOB = 0 OR 1 SO THAT
C                                THE MATRIX A HAS ALREADY BEEN FACTORED.
C                                IN THIS CASE, OUTPUT MATRICES A AND XL
C                                MUST HAVE BEEN SAVED FOR REUSE IN THE
C                                CALL TO BANDMX.
C                 XL       - WORK AREA OF DIMENSION N*(NLC+1). THE FIRST
C                              NLC*N LOCATIONS OF XL CONTAIN COMPONENTS OF
C                              THE L MATRIX OF THE L-U DECOMPOSITION OF A
C                              ROWWISE PERMUTATION OF A. THE LAST N
C                              LOCATIONS CONTAIN THE PIVOT INDICES.
C                 IER      - ERROR PARAMETER. (OUTPUT)
C                            TERMINAL ERROR
C                              IER = 129 INDICATES THAT MATRIX A IS
C                                ALGORITHMICALLY SINGULAR. (SEE THE
C                                CHAPTER L PRELUDE).
C   REQUIRED SUBROUTINES-SUB1,SUB2
C-----------------------------------------------------------------------
      SUBROUTINE BANDMX (N,NLC,NUC,IA,M,IB,IJOB,IER)
      COMMON/M3/  A(10004,205),B(10004,5),XL(10004,105)
C                 DIMENSION A(Y*X,NUC+NLC+1)    2-D
C                           XL(Y*X,(NLC+1))     1-D
C                           B(Y*X,1)            2-D
      DATA                 ZERO/0./,ONE/1.0/
C                                  FIRST EXECUTABLE STATEMENT
      IER = 0
      JBEG = NLC+1
      NLC1 = JBEG
      IF (IJOB .EQ. 2) GO TO 80
      RN = N
C                                  RESTRUCTURE THE MATRIX
C                                  FIND RECIPROCAL OF THE LARGEST
C                                  ABSOLUTE VALUE IN ROW I
      I = 1
      NC = JBEG+NUC
      NN = NC
      JEND = NC
      IF (N .EQ. 1 .OR. NLC .EQ. 0) GO TO 25
    5 K = 1
      P = ZERO
      DO 10 J = JBEG,JEND
         A(I,K) = A(I,J)
         Q =   ABS(A(I,K))
         IF (Q .GT. P) P = Q
         K = K+1
   10 CONTINUE
      IF (P .EQ. ZERO) GO TO 135
      XL(I,NLC1) = ONE/P
      IF (K .GT. NC) GO TO 20
      DO 15 J = K,NC
         A(I,J) = ZERO
   15 CONTINUE
   20 I = I+1
      JBEG = JBEG-1
      IF (JEND-JBEG .EQ. N) JEND = JEND-1
      IF (I .LE. NLC) GO TO 5
      JBEG = I
      NN = JEND
   25 JEND = N-NUC
      DO 40 I = JBEG,N
         P = ZERO
         DO 30 J = 1,NN
            Q =  ABS(A(I,J))
            IF (Q .GT. P) P = Q
   30    CONTINUE
         IF (P .EQ. ZERO) GO TO 135
         XL(I,NLC1) = ONE/P
         IF (I .EQ. JEND) GO TO 37
         IF (I .LT. JEND) GO TO 40
         K = NN+1
         DO 35 J = K,NC
            A(I,J) = ZERO
   35    CONTINUE
   37    NN = NN-1
   40 CONTINUE
      L = NLC
C                                        L-U DECOMPOSITION
      DO 75 K = 1,N
         P =  ABS(A(K,1))*XL(K,NLC1)
         I = K
         IF (L .LT. N) L = L+1
         K1 = K+1
         IF (K1 .GT. L) GO TO 50
         DO 45 J = K1,L
            Q = ABS(A(J,1))*XL(J,NLC1)
            IF (Q .LE. P) GO TO 45
            P = Q
            I = J
   45    CONTINUE
   50    XL(I,NLC1) = XL(K,NLC1)
         XL(K,NLC1) = I
C                                        SINGULARITY FOUND
         Q = RN+P
         IF (Q .EQ. RN) GO TO 135
C                                        INTERCHANGE ROWS I AND K
         IF (K .EQ. I) GO TO 60
         DO 55 J = 1,NC
            P = A(K,J)
            A(K,J) = A(I,J)
            A(I,J) = P
   55    CONTINUE
   60    IF (K1 .GT. L) GO TO 75
         DO 70 I = K1,L
            P = A(I,1)/A(K,1)
            IK = I-K
            XL(K1,IK) = P
            DO 65 J = 2,NC
               A(I,J-1) = A(I,J)-P*A(K,J)
   65       CONTINUE
            A(I,NC) = ZERO
   70    CONTINUE
   75 CONTINUE
      IF (IJOB .EQ. 1) GO TO 9005
C                                        FORWARD SUBSTITUTION
   80 L = NLC
      DO 105 K = 1,N
         I = XL(K,NLC1)
         IF (I .EQ. K) GO TO 90
         DO 85 J = 1,M
            P = B(K,J)
            B(K,J) = B(I,J)
            B(I,J) = P
   85    CONTINUE
   90    IF (L .LT. N) L = L+1
         K1 = K+1
         IF (K1 .GT. L) GO TO 105
         DO 100 I = K1,L
            IK = I-K
            P = XL(K1,IK)
            DO 95 J = 1,M
               B(I,J) = B(I,J)-P*B(K,J)
   95       CONTINUE
  100    CONTINUE
  105 CONTINUE
C                                        BACKWARD SUBSTITUTION
```

```
      JBEG = NUC+NLC
      DO 125 J = 1,M
         L = 1
         K1 = N+1
         DO 120 I = 1,N
            K = K1-I
            P = B(K,J)
            IF (L .EQ. 1) GO TO 115
            DO 110 KK = 2,L
               IK = KK+K
               P = P-A(K,KK)*B(IK-1,J)
  110       CONTINUE
  115       B(K,J) = P/A(K,1)
            IF (L .LE. JBEG) L = L+1
  120    CONTINUE
  125 CONTINUE
      GO TO 9005
  135 IER = 129
 9000 CONTINUE
      WRITE(6,998)
  998 FORMAT(1X,'CALL SUB1')
      CALL SUB1(IER,'LEQT1B  ')
 9005 RETURN
      END
C
C
C
C   SUB1
C
C           --------------------------------------------------------------
C PURPOSE              - PRINT A MESSAGE REFLECTING AN ERROR CONDITION
C USAGE                - CALL SUB1 (IER,NAME)
C ARGUMENTS     IER    - ERROR PARAMETER. (INPUT)
C                        IER = I+J WHERE
C                          I = 128 IMPLIES TERMINAL ERROR,
C                          I =  64 IMPLIES WARNING WITH FIX, AND
C                          I =  32 IMPLIES WARNING.
C                          J = ERROR CODE RELEVANT TO CALLING
C                              ROUTINE.
C               NAME   - A SIX CHARACTER LITERAL STRING GIVING THE
C                        NAME OF THE CALLING ROUTINE. (INPUT)
C REQUIRED SUBRTINE= SUB2
C REMARKS       THE ERROR MESSAGE PRODUCED BY SUB1 IS WRITTEN
C               ONTO THE STANDARD OUTPUT UNIT. THE OUTPUT UNIT
C               NUMBER CAN BE DETERMINED BY CALLING SUB2 AS
C               FOLLOWS..   CALL SUB2(1,NIN,NOUT).
C               THE OUTPUT UNIT NUMBER CAN BE CHANGED BY CALLING
C               SUB2 AS FOLLOWS..
C                                NIN = 0
C                                NOUT = NEW OUTPUT UNIT NUMBER
C                                CALL SUB2(3,NIN,NOUT)
C               SEE THE UGETIO DOCUMENT FOR MORE DETAILS.
C         ------------------------------------------------------------
      SUBROUTINE SUB1(IER,NAME)
C                                  SPECIFICATIONS FOR ARGUMENTS
      INTEGER            IER
      REAL*8         NAME
C                                  SPECIFICATIONS FOR LOCAL VARIABLES
      REAL*8         NAMSET,NAMEQ
      DATA               NAMSET/6HUERSET  /
      DATA               NAMEQ/6H         /
C                                  FIRST EXECUTABLE STATEMENT
      DATA               LEVEL/4/,IEQDF/0/,IEQ/1H=/
      IF (IER.GT.999) GO TO 25
      IF (IER.LT.-32) GO TO 55
      IF (IER.LE.128) GO TO 5
      IF (LEVEL.LT.1) GO TO 30
C                                  PRINT TERMINAL MESSAGE
      CALL SUB2(1,NIN,IOUNIT)
      IF (IEQDF.EQ.1) WRITE(IOUNIT,35) IER,NAMEQ,IEQ,NAME
      IF (IEQDF.EQ.0) WRITE(IOUNIT,35) IER,NAME
      GO TO 30
    5 IF (IER.LE.64) GO TO 10
      IF (LEVEL.LT.2) GO TO 30
C                                  PRINT WARNING WITH FIX MESSAGE
      CALL SUB2(1,NIN,IOUNIT)
      IF (IEQDF.EQ.1) WRITE(IOUNIT,40) IER,NAMEQ,IEQ,NAME
      IF (IEQDF.EQ.0) WRITE(IOUNIT,40) IER,NAME
      GO TO 30
   10 IF (IER.LE.32) GO TO 15
C                                  PRINT WARNING MESSAGE
      IF (LEVEL.LT.3) GO TO 30
      CALL SUB2(1,NIN,IOUNIT)
      IF (IEQDF.EQ.1) WRITE(IOUNIT,45) IER,NAMEQ,IEQ,NAME
      IF (IEQDF.EQ.0) WRITE(IOUNIT,45) IER,NAME
      GO TO 30
   15 CONTINUE
C                                   CHECK FOR UERSET CALL
      IF (NAME.NE.NAMSET) GO TO 25
      LEVOLD = LEVEL
      LEVEL = IER
      IER = LEVOLD
      IF (LEVEL.LT.0) LEVEL = 4
      IF (LEVEL.GT.4) LEVEL = 4
      GO TO 30
   25 CONTINUE
      IF (LEVEL.LT.4) GO TO 30
C                                   PRINT NON-DEFINED MESSAGE
      CALL SUB2(1,NIN,IOUNIT)
      IF (IEQDF.EQ.1) WRITE(IOUNIT,50) IER,NAMEQ,IEQ,NAME
      IF (IEQDF.EQ.0) WRITE(IOUNIT,50) IER,NAME
   30 IEQDF = 0
      GO TO 65
   35 FORMAT(19H *** TERMINAL ERROR,10X,7H(IER = ,I3,
     &20H) FROM IMSL ROUTINE ,A6,A1,A6)
   40 FORMAT(36H *** WARNING WITH FIX ERROR  (IER = ,I3,
     &20H) FROM IMSL ROUTINE ,A6,A1,A6)
   45 FORMAT(18H *** WARNING ERROR,11X,7H(IER = ,I3,
     &20H) FROM IMSL ROUTINE ,A6,A1,A6)
   50 FORMAT(20H *** UNDEFINED ERROR,9X,7H(IER = ,I5,
     &20H) FROM IMSL ROUTINE ,A6,A1,A6)
C                                   SAVE P FOR P = R CASE
C                                     P IS THE PAGE NAME
C                                     R IS THE ROUTINE NAME
   55 IEQDF = 1
      NAMEQ = NAME
   65 CONTINUE
      RETURN
      END
C
C
C
C    SUB2
C-----------------------------------------------------------------------
C     PURPOSE             - TO RETRIEVE CURRENT VALUES AND TO SET NEW
C                             VALUES FOR INPUT AND OUTPUT UNIT
C                             IDENTIFIERS.
C
C     USAGE               - CALL SUB2(IOPT,NIN,NOUT)
C
C     ARGUMENTS    IOPT   - OPTION PARAMETER. (INPUT)
C                             IF IOPT=1, THE CURRENT INPUT AND OUTPUT
C                             UNIT IDENTIFIER VALUES ARE RETURNED IN NIN
C                             AND NOUT, RESPECTIVELY.
C                             IF IOPT=2 (3) THE INTERVAL VALUE OF
C                             NIN (NOUT) IS RESET FOR SUBSEQUENT USE.
C                  NIN    - INPUT UNIT IDENTIFIER.
C                             OUTPUT IF IOPT=1, INPUT IF IOPT=2.
C                  NOUT   - OUTPUT UNIT IDENTIFIER.
C                             OUTPUT IF IOPT=1, INPUT IF IOPT=3.
C     REMARKS      EACH IMSL ROUTINE THAT PERFORMS INPUT AND/OR OUTPUT
C                  OPERATIONS CALLS SUB2 TO OBTAIN TXE CURRENT UNIT
C                  IDENTIFIER VALUES. IF SUB2 IS CALLED WITH IOPT=2 OR 3
C                  NEW UNIT IDENTIFIER VALUES ARE ESTABLISHED. SUBSEQUENT
C                  INPUT/OUTPUT IS PERFORMED ON THE NEW UNITS.
C-----------------------------------------------------------------------
C
      SUBROUTINE SUB2(IOPT,NIN,NOUT)
C                                  SPECIFICATIONS FOR ARGUMENTS
      IMPLICIT INTEGER*4(I-N)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      INTEGER            IOPT,NIN,NOUT
C                                  SPECIFICATIONS FOR LOCAL VARIABLES
      INTEGER            NIND,NOUTD
      DATA               NIND/5/,NOUTD/6/
C                                  FIRST EXECUTABLE STATEMENT
      IF (IOPT.EQ.3) GO TO 10
      IF (IOPT.EQ.2) GO TO 5
      IF (IOPT.NE.1) GO TO 9005
      NIN = NIND
      NOUT = NOUTD
      GO TO 9005
    5 NIND = NIN
      GO TO 9005
   10 NOUTD = NOUT
 9005 RETURN
      END
```

Table A1. Sample output from SSCONDUCT—provides initial information and final temperature distribution.

```
   DATA FOR THIS RUN OF SSCONDUCT:

X= 21
Y= 31
A=  1
NISO=   4
DS=   0.100
DI=  50.0000
TISO(B),B=1,NISO:
 100.00
  70.00
  50.00
  30.00
THK(L),L=1,A:
   1.4400
H(L),L=1,A:
   0.0000
FI(1)=    0.0000
FI(2)=    0.0000
FI(3)=    0.0000
FI(4)=    0.0000
RAY(I,J,3):
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
   1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00    1.00
```

```
..THE REST OF THE COLUMNS:
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
     1.00      1.00      1.00      1.00
 RAY(I,J,2):
    11.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
     6.00     15.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00
    14.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00
```

```
..THE REST OF THE COLUMNS:
   2.00      2.00      2.00     11.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      5.00
   1.00      1.00      5.00      5.00
   1.00      1.00      1.00      5.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   1.00      1.00      1.00      3.00
   8.00      8.00      8.00     19.00
   6.00      6.00      6.00     18.00
RAY(I,J,1):
  10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
  10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00
```

```
..THE REST OF THE COLUMNS:
   10.00   10.00   10.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00  100.00
    0.00    0.00  100.00  100.00
    0.00    0.00    0.00  100.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
    0.00    0.00    0.00   10.00
   10.00   10.00   10.00   10.00
BOUNDARY CONDITIONS:
CONSTANT CORNER RAY( 1, 1,1)= 10.00
CONSTANT CORNER RAY( 1,21,1)= 10.00
CONSTANT INTERIOR NODE RAY(10,21,1)=100.00
CONSTANT INTERIOR NODE RAY(11,20,1)=100.00
CONSTANT INTERIOR NODE RAY(11,21,1)=100.00
CONSTANT INTERIOR NODE RAY(12,21,1)=100.00


BOTTOM LEFT CORNER SEMI-INFINITE
TOP BOUNDARY      CONSTANT TEMPERATURE
LEFT BOUNDARY     SEMI-INFINITE
BOTTOM BOUNDARY SEMI-INFINITE
RIGHT BOUNDARY   CONSTANT HEAT FLUX

BANDWIDTH= 43
 NLC, NUC,   N,   IA,    M,  IZ,IJOB, IER:
   21   21  651  651    1  651    0  478

IER=  0:
    IF IER=129, MATRIX IS SINGULAR
    IF IER=0, EVERYTHING IS OKAY.
```

```
FINAL TEMPERATURE DISTRIBUTION:
   10.00  10.00  10.00  10.00  10.00  10.00  10.00  10.00  10.00  10.00  10.00  10.00  10.00  10.00  10.00  10.00  10.00
   10.04  11.92  11.97  12.05  12.14  12.26  12.40  12.57  12.77  12.99  13.24  13.53  13.85  14.19  14.55  14.92  15.28
   10.08  13.61  13.93  14.08  14.27  14.50  14.78  15.12  15.50  15.95  16.46  17.03  17.66  18.36  19.09  19.85  20.60
   10.12  15.67  15.84  16.06  16.35  16.69  17.11  17.60  18.18  18.84  19.60  20.46  21.42  22.48  23.62  24.80  25.98
   10.16  17.48  17.70  17.99  18.36  18.82  19.36  20.01  20.77  21.64  22.65  23.80  25.09  26.53  28.09  29.75  31.44
   10.19  19.21  19.48  19.84  20.29  20.85  21.52  22.31  23.24  24.32  25.56  26.99  28.61  30.44  32.47  34.68  36.99
   10.23  20.86  21.17  21.59  22.12  22.77  23.55  24.47  25.56  26.82  28.29  29.98  31.93  34.16  36.69  39.50  42.56
   10.26  22.40  22.76  23.23  23.83  24.56  25.43  26.47  27.69  29.12  30.78  32.72  34.97  37.57  40.59  44.07  48.02
   10.29  23.84  24.23  24.75  25.40  26.20  27.16  28.29  29.63  31.19  33.01  35.15  37.64  40.58  44.05  48.17  53.08
   10.32  25.15  25.58  26.13  26.83  27.69  28.71  29.91  31.33  32.99  34.93  37.21  39.88  43.06  46.86  51.48  57.22
   10.34  26.33  26.79  27.38  28.11  29.09  30.07  31.32  32.79  34.51  36.52  38.87  41.63  44.91  48.85  53.68  59.78
   10.37  27.39  27.86  28.47  29.23  30.15  31.24  32.52  34.01  35.74  37.76  40.11  42.86  46.09  49.94  54.60  60.33
   10.38  28.31  28.80  29.42  30.20  31.12  32.22  33.50  34.98  36.70  38.68  40.96  43.60  46.67  50.24  54.43  59.36
   10.40  29.09  29.59  30.23  31.01  31.93  33.02  34.27  35.72  37.38  39.28  41.45  43.92  46.73  49.92  53.54  57.57
   10.41  29.75  30.26  30.90  31.67  32.58  33.64  34.86  36.25  37.84  39.62  41.64  43.89  46.41  49.19  52.22  55.44
   10.42  30.28  30.80  31.43  32.19  33.08  34.10  35.27  36.60  38.08  39.74  41.58  43.61  45.82  48.20  50.71  53.27
   10.42  30.70  31.22  31.84  32.58  33.44  34.42  35.53  36.77  38.16  39.68  41.34  43.15  45.07  47.09  49.16  51.19
   10.42  31.01  31.53  32.14  32.85  33.68  34.61  35.66  36.81  38.09  39.47  40.97  42.56  44.22  45.93  47.63  49.26
   10.42  31.23  31.73  32.33  33.02  33.81  34.69  35.67  36.74  37.91  39.16  40.50  41.89  43.33  44.78  46.19  47.51
   10.41  31.35  31.85  32.43  33.09  33.84  34.67  35.58  36.57  37.64  38.77  39.96  41.19  42.43  43.66  44.84  45.92
   10.39  31.39  31.88  32.45  33.08  33.79  34.57  35.42  36.34  37.31  38.33  39.39  40.46  41.54  42.60  43.59  44.48
   10.38  31.36  31.84  32.39  33.00  33.67  34.41  35.20  36.04  36.92  37.85  38.79  39.74  40.68  41.59  42.44  43.19
   10.35  31.27  31.74  32.27  32.86  33.49  34.18  34.92  35.69  36.51  37.34  38.19  39.03  39.86  40.65  41.37  42.01
   10.33  31.12  31.59  32.10  32.56  33.26  33.91  34.60  35.32  36.06  36.82  37.59  38.35  39.09  39.77  40.40  40.95
   10.30  30.93  31.38  31.88  32.42  32.99  33.61  34.25  34.92  35.61  36.30  37.00  37.68  38.34  38.95  39.51  39.99
   10.27  30.69  31.13  31.62  32.13  32.63  33.27  33.87  34.50  35.14  35.78  36.42  37.04  37.64  38.19  38.69  39.11
   10.23  30.41  30.85  31.32  31.82  32.35  32.90  33.48  34.07  34.67  35.27  35.86  36.43  36.98  37.48  37.93  38.32
   10.19  30.11  30.54  30.99  31.48  31.99  32.52  33.07  33.63  34.20  34.76  35.32  35.85  36.36  36.83  37.24  37.60
   10.14  29.77  30.19  30.64  31.11  31.60  32.12  32.65  33.18  33.73  34.27  34.80  35.30  35.78  36.22  36.62  36.95
   10.09  29.41  29.82  30.26  30.72  31.20  31.70  32.21  32.74  33.26  33.78  34.29  34.78  35.24  35.67  36.04  35.37
   10.04  10.06  10.08  10.10  10.12  10.14  10.16  10.17  10.19  10.20  10.21  10.23  10.24  10.25  10.25  10.26  10.27

..THE REST OF THE COLUMNS:
   10.00  10.00  10.00  10.00
   15.61  15.87  16.04  16.10
   21.28  21.83  22.20  22.33
   27.07  27.98  28.59  28.80
   33.05  34.43  35.37  35.72
   39.27  41.30  42.76  43.31
   45.74  48.74  51.07  52.01
   52.38  56.87  60.75  62.60
   58.90  65.59  72.47  76.88
   64.54  74.13  86.65 100.00
   67.91  79.74 100.00 100.00
   67.51  76.92  88.76 100.00
   65.12  71.62  78.11  82.00
   61.93  66.34  70.06  71.76
   58.71  61.74  64.03  64.94
   55.73  57.87  59.38  59.94
   53.07  54.63  55.68  56.06
   50.72  51.90  52.67  52.94
   48.66  49.57  50.15  50.35
   46.85  47.56  48.02  48.17
   45.24  45.82  46.18  46.31
   43.82  44.29  44.59  44.69
   42.54  42.94  43.19  43.27
   41.41  41.74  41.95  42.02
   40.38  40.68  40.86  40.92
   39.46  39.72  39.88  39.93
   38.63  38.87  39.01  39.06
   37.89  38.10  38.23  38.27
   37.22  37.42  37.54  37.58
   36.62  36.81  36.93  36.97
   10.27  10.28  10.28  10.28
```

Table A2. Sample output from SSCONDUCT— isotherm locations.

```
      72
ISOTHM #  1 HAS    4 POINTS
             1.900             1.000
             2.000             0.900
             2.000             1.000
             2.000             1.100
ISOTHM #  2 HAS   12 POINTS
             1.864             0.800
             1.900             0.779
             1.757             0.900
             1.800             0.852
             1.718             1.000
             1.726             1.100
             1.775             1.200
             1.898             1.300
             1.800             1.231
             1.900             1.301
             2.000             0.752
             2.000             1.326
ISOTHM #  3 HAS   27 POINTS
             1.854             0.600
             1.900             0.587
             1.645             0.700
             1.700             0.664
             1.800             0.615
             1.537             0.800
             1.600             0.739
             1.468             0.900
             1.500             0.855
             1.424             1.000
             1.401             1.100
             1.393             1.200
             1.400             1.119
             1.402             1.300
             1.400             1.276
             1.427             1.400
             1.472             1.500
             1.542             1.600
             1.500             1.546
             1.651             1.700
             1.600             1.662
             1.700             1.735
             1.874             1.800
             1.800             1.781
             1.900             1.807
             2.000             0.577
             2.000             1.916
ISOTHM #  4 HAS   29 POINTS
             1.515             0.400
             1.600             0.374
             1.700             0.349
             1.800             0.331
             1.900             0.321
             1.276             0.500
             1.300             0.489
             1.400             0.444
             1.500             0.405
             1.101             0.600
             1.200             0.542
             0.953             0.700
             1.000             0.669
             1.100             0.601
             0.824             0.800
             0.900             0.743
             0.706             0.900
             0.800             0.822
             0.594             1.000
             0.600             0.995
             0.700             0.906
             0.484             1.100
             0.500             1.087
             0.375             1.200
             0.400             1.180
             0.264             1.300
             0.300             1.271
             0.200             1.361
             2.000             0.317
```

Table A3. Sample input to SSCONDUCT—*file generated by* SSDATA (subroutine of SSCONDUCT).

```
  0.1000  50.0000
1 21 31   4
  1.4400
  0.0000
100.0000
 70.0000
 50.0000
 30.0000
  0.0030   0.0000   0.0000   0.0000   1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
   1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00     1.00
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      1.00      1.00
     11.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00
      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00      2.00     11.00      6.00
      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      3.00      6.00      7.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      3.00      6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      3.00      6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      5.00      6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      5.00      5.00
      6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      5.00      6.00
      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      3.00      6.00      7.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      3.00      6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      3.00      6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      3.00      6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      3.00
      6.00      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      3.00      6.00
      7.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      3.00      6.00      7.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00      1.00
      1.00      1.00      1.00      3.00      6.00     15.00      8.00      8.00      8.00      8.00      8.00
      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00      8.00
      8.00      8.00     19.00     14.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00
      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00      6.00
      6.00     18.00
     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00
     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00     10.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00     10.00     10.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00     10.00     10.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00     10.00     10.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00     10.00     10.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00     10.00     10.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00     10.00     10.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00     10.00     10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00     10.00     10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
    100.00     10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00    100.00    100.00
     10.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00
      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00      0.00    100.00     10.00
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 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00      10.00      10.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00      10.00      10.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00      10.00      10.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00      10.00      10.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00      10.00      10.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00      10.00      10.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00      10.00      10.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00      10.00      10.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
10.00      10.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00      10.00
10.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00      10.00      10.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00      10.00      10.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00      10.00      10.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00      10.00      10.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00      10.00      10.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00      10.00      10.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00      10.00      10.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00       0.00
 0.00       0.00      10.00      10.00      10.00      10.00      10.00      10.00      10.00      10.00      10.00
10.00      10.00      10.00      10.00      10.00      10.00      10.00      10.00      10.00      10.00      10.00
10.00      10.00
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APPENDIX B: FINITE ELEMENT PROGRAM — DOCUMENTATION AND SAMPLE INPUT AND OUTPUT

Program FEHEAT

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C        THIS PROGRAM SOLVES 2 DIMENSIONAL STEADY STATE HEAT TRANSFER PROBLEMS
C        USING THE FINITE ELEMENT METHOD.  (ONLY TRIANGULAR ELEMENTS MAY BE USED)
C
C        DIMENSION THE MATRICES WHICH WILL NOT BE STORED IN COMMON STORAGE
      DIMENSION NMJ(300),NEC(300)
      DIMENSION BT(175)
C        DIMENSION THE REMAINING MATRICES INTO BLANK COMMON BLOCK STORAGE.
      COMMON/M1I/IMAT(300)
      COMMON/M1R/TKM(300),QI(300)
      COMMON/M12I/NC1(300),NC2(300)
      COMMON/M12R/X(175),Y(175),H(300),FQIE(175)
      COMMON/M2I/MC1,MC2,NN,NCASE,NBHF(175)
      COMMON/M2R/BHF(175),TA(300),RHST(175),RHS(300)
      COMMON/M23I/MC,NBT(175)
      COMMON/M23R/R1(175)
      COMMON/M3I/NE,NCN,NDF,NSZF,NBAND
      COMMON/M3R/TM(30,175)
      COMMON/M13I/NODE(300,3)
      COMMON/M13R/TME(3,3)
C        OPEN THE NECESSARY DATA FILES.
      CALL CONTRL(1,'NPDXXX',5)
      CALL CONTRL(1,'EDXXXX',6)
      CALL CONTRL(1,'BCTXXX',7)
      CALL CONTRL(1,'IHGXXX',8)
      CALL CONTRL(1,'SHFXXX',9)
      CALL CONTRL(1,'CHSXXX',14)
      CALL CONTRL(1,'QUANXX',11)
      CALL CONTRL(2,'OUTPUT',12)
      CALL CONTRL(1,'TIOTXX',13)
      READ(11,107,END=1005)NN,NE,MC,MJC,MC1,MC2,NIT
1005  CONTINUE
C        INITILIZE ALL MATRICES AND PARAMETERS TO ZERO.
      NCN=3
      NDF=1
      NCASE=1
      NSZF=NN*NDF
      TKM(1)=.833
      DO 2 I=1,NN,1
      X(I)=0.0
      Y(I)=0.0
      NBT(I)=0
      BT(I)=0.0
      NBHF(I)=0
      BHF(I)=0.0
      RHS(I)=0.0
      FQIE(I)=0.0
2        CONTINUE
      DO 28 J=1,NE,1
      NC1(J)=0
      NC2(J)=0
      H(J)=0.0
      TA(J)=0.0
      IMAT(J)=0
      NMJ(J)=0
      QI(J)=0.0
28         CONTINUE
C        READ IN THE NODAL POINT DATA.
      DO 3 J=1,NN,1
      READ(5,101,END=1000) I,X(I),Y(I)
3        CONTINUE
1000       CONTINUE
C        READ IN THE ELEMENT DATA.
      DO 4 I=1,NE,1
      READ(6,102,END=1001) J,NODE(J,1),NODE(J,2),NODE(J,3),IMAT(J)
4        CONTINUE
1001  CONTINUE
C        READ IN THE FIXED TEMPERATURE BOUNDARY CONDITIONS.
      IF(MC.EQ.0) GO TO 5
```

```
        DO 6 IM=1,MC,1
        READ(7,103,END=1002) M,NBT(M),BT(NBT(M))
6          CONTINUE
1002      CONTINUE
C          READ IN THE INTERNAL HEAT GENERATION VALUES FOR EACH ELEMENT.
5          IF(MJC.EQ.0) GO TO 1
        DO 7 IMJ=1,MJC,1
        READ(8,103) MJ,NMJ(MJ),QI(NMJ(MJ))
7          CONTINUE
C          READ IN THE HEAT FLUX VALUES FOR NODES WHERE HEAT FLUX IS SPECIFIED.
1          IF(MC1.EQ.0) GO TO 8
        DO 9 IMC1=1,MC1,1
        READ(9,103) M1,NBHF(M1),BHF(NBHF(M1))
9          CONTINUE
C          FOR BOUNDARY SEGMENTS SUBJECT TO CONVECTIVE HEAT TRANSFER READ
C          IN CORRESPONDING NODES ,H,AND AMBIENT TEMPERATURE.
8          IF(MC2.EQ.0) GO TO 10
        DO 11 M2=1,MC2,1
        READ(14,106) NEC(M2),NC1(NEC(M2)),NC2(NEC(M2)),H(NEC(M2)),
       X TA(NEC(M2))
11           CONTINUE
10           CONTINUE
        WRITE(12,120)NN,NE
        WRITE(12,121)
        DO 81 I=1,NN,1
        WRITE(12,101)I,X(I),Y(I)
81           CONTINUE
        WRITE(12,122)
        DO 82 J=1,NE,1
        WRITE(12,102)J,NODE(J,1),NODE(J,2),NODE(J,3),IMAT(J)
82           CONTINUE
        IF(MC)93,93,91
91           WRITE(12,123)
        DO 83 M=1,MC,1
        WRITE(12,103)M,NBT(M),BT(NBT(M))
83           CONTINUE
93           CONTINUE
        IF(MJC)94,94,92
92           WRITE(12,124)
        DO 84 I=1,MJC,1
        WRITE(12,103)MJ,NMJ(MJ),QI(NMJ(MJ))
84           CONTINUE
94           CONTINUE
        IF(MC1)95,95,97
97           WRITE(12,125)
        DO 85 M1=1,MC1,1
        WRITE(12,103)M1,NBHF(M1),BHF(NBHF(M1))
85           CONTINUE
95           CONTINUE
        IF(MC2)96,96,98
98           WRITE(12,126)
        DO 86 M2=1,MC2,1
        WRITE(12,126)NEC(M2),NC1(NEC(M2)),NC2(NEC(M2)),H(NEC(M2)),
       X TA(NEC(M2))
86           CONTINUE
96           CONTINUE
C          FIND BANDWIDTH FOR TRIANGULAR ELEMENTS.
        MX=0
        DO 66 I=1,NE,1
        L1=IABS(NODE(I,1)-NODE(I,3))
        L2=IABS(NODE(I,3)-NODE(I,2))
        L3=IABS(NODE(I,2)-NODE(I,1))
         MX=MAX0(MX,L1,L2,L3)
66      CONTINUE
        NBAND=NDF*(MX+1)
        WRITE(12,129)NBAND
        MUD=NBAND-1
        IO=0
        IOP=1
        CALL FORMK
        CALL FRHS([illegible]T)
        CALL MCHB(RHS,IM,NSZF,1,MUD,IOP,0.5E-6,IER,IO)
        WRITE(12,127)
        DO 200 I=1,NN,1
        WRITE(12,130)I,RHS(I)
200           CONTINUE
        IF(NIT.EQ.0) GO TO 228
        CALL ISOTHM(RHS,NODE,X,Y,NE,NIT)
228           CONTINUE
101           FORMAT(I5,2F10.4)
102           FORMAT(I5,4I6)
103           FORMAT(I5,I6,F10.4)
106           FORMAT(3I6,2F10.4)
107           FORMAT(7I6)
```

```
109         FORMAT(5X,'THE BANDWIDTH IS',I6)
120         FORMAT(//,5X,'TOTAL NUMBER OF NODES =',I6,3X,
     X   'TOTAL NUMBER OF ELEMENTS = ',I6)
121         FORMAT(//,2X,'NODE',3X,'GLOBAL',4X,'GLOBAL',/,3X,'NO.',6X,'X',
     X 9X,'Y',1X,/,'--------------------------')
122         FORMAT(//,1X,'ELEMENT',1X,'NODE',2X,'NODE',2X,'NODE',2X,
     X  'MATL.',/,3X,'NO.',4X,'1',5X,'2',5X,'3',4X,'TYPE',/,1X,'--------',
     X  '------------------------')
123         FORMAT(//,3X,'B.C.',2X,'NODE',2X,'TEMP',/,4X,'NO.',3X,'NO.',2X,
     X  '(F)',/,1X,'------------------------')
129         FORMAT(//,2X,'B.C.',2X,'NODE',6X,'QI',/,2X,'NO.',3X,'NO.',5X,
     X  '(UNITS)')
125         FORMAT(//,2X,'B.C',2X,'NODE',6X,'HF',/,2X,'NO.',3X,'NO.',5X,
     X  '(UNITS)')
126         FORMAT(//,1X,'SEGMENT',1X,'NODE',2X,'NODE',4X,'H',3X,'AMB',/,3X,
     X   'NO.',4X,'1',5X,'2',3X,'(UNITS)',2X,'TEMP(F)')
127         FORMAT(//,13X,'NODE',2X,'TEMPERATURE (F)',
     X   /,12X,'------------------------')
130         FORMAT(10X,I6,F13.2)
      CALL CONTRL(4,'NPDXXX',5)
      CALL CONTRL(4,'EDXXXX',6)
      CALL CONTRL(4,'BCTXXX',7)
      CALL CONTRL(4,'IHGXXX',8)
      CALL CONTRL(4,'SHFXXX',9)
      CALL CONTRL(4,'CBSXXX',14)
      CALL CONTRL(4,'QUANXX',11)
      CALL CONTRL(4,'OUTPUT',12)
      CALL CONTRL(4,'TIOTXX',13)
      CALL EXIT
      END
C     ------------------------------------------------------------------
C
      SUBROUTINE TSM(K)
      DIMENSION XC(3),YC(3),TK(300)
      COMMON/M1I/IMAT(300)
      COMMON/M1R/TKM(300),QI(300)
      COMMON/M12I/NC1(300),NC2(300)
      COMMON/M12R/X(175),Y(175),H(300),FQIE(175)
      COMMON/M13I/NODE(300,3)
      COMMON/M13R/TME(3,3)
C        K IS THE ELEMENT NUMBER.
      TK(K)=TKM(IMAT(K))
      N1=NODE(K,1)
      N2=NODE(K,2)
      N3=NODE(K,3)
C        DEFINE THE ELEMENT NODAL X AND Y GLOBAL COORDINATES.
      XC(1)=X(N1)
      YC(1)=Y(N1)
      XC(2)=X(N2)
      YC(2)=Y(N2)
      XC(3)=X(N3)
      YC(3)=Y(N3)
      A=1.0
      AA=1.0
C        DEFINE THE A'S,B'S,AND C'S OF THE LINEAR INTERPOLATION FUNCTIONS.
      B1=YC(2)-YC(3)
      B2=YC(3)-YC(1)
      B3=YC(1)-YC(2)
      C1=XC(3)-XC(2)
      C2=XC(1)-XC(3)
      C3=XC(2)-XC(1)
C        DETERMINE THE ELEMENT AREA
      DEL=ABS(0.5*(XC(1)*(YC(2)-YC(3))+XC(2)*(YC(3)-YC(1))+XC(3)
     X *(YC(1)-YC(2))))
C        FORM THE INFLUENCE MATRIX FOR TEMPERATURE.
      CONST=(TK(K)*A/(4.0*DEL))*AA
      TME(1,1)=(B1**2+C1**2)*CONST
      TME(1,2)=(B1*B2+C1*C2)*CONST
      TME(1,3)=(B1*B3+C1*C3)*CONST
      TME(2,1)=TME(1,2)
      TME(2,2)=(B2**2+C2**2)*CONST
      TME(2,3)=(B2*B3+C2*C3)*CONST
      TME(3,1)=TME(1,3)
      TME(3,2)=TME(2,3)
      TME(3,3)=(B3**2+C3**2)*CONST
      IF(NC1(K))315,330,340
340      KK1=NC1(K)
      KK2=NC2(K)
      IF(KK1.EQ.N1) K1=1
      IF(KK1.EQ.N2) K1=2
      IF(KK1.EQ.N3) K1=3
      IF(KK2.EQ.N1) K2=1
      IF(KK2.EQ.N2) K2=2
      IF(KK2.EQ.N3) K2=3
      IF(K1.EQ.K2) GO TO 315
      CONSTC=H(K)*(SQRT((X(NC1(K))-X(NC2(K)))**2+(Y(NC1(K))-
     X Y(NC2(K)))**2))
      TME(K1,K1)=TME(K1,K1)+CONSTC/3.
      TME(K1,K2)=TME(K1,K2)+CONSTC/6.
      TME(K2,K2)=TME(K2,K2)+CONSTC/3.
      TME(K2,K1)=TME(K2,K1)+CONSTC/6.
330       CONTINUE
C       FORM INFLUENCE MATRIX FOR INTERNAL HEAT GENERATION.
      QIE=QI(K)
      CONSTQ=DEL/12.
      QN=4.*QIE*CONSTQ
      FQIE(N1)=FQIE(N1)+QN
      FQIE(N2)=FQIE(N2)+QN
      FQIE(N3)=FQIE(N3)+QN
      GO TO 10
315       WRITE(12,320)
320       FORMAT(20X,'*******ERROR******')
10       CONTINUE
      RETURN
      END
C       -------------------- ------------ ---------------------------------
      SUBROUTINE FRHS(BT)
      DIMENSION BL(300),RBL(300)
      DIMENSION BT(175)
      COMMON/M12I/NC1(300),NC2(300)
      COMMON/M12R/X(175),Y(175),H(300),FQIE(175)
      COMMON/M2I/MC1,MC2,NN,NCASE,NBHF(175)
      COMMON/M2R/BHF(175),TA(300),RHST(175),RHS(300)
      COMMON/M23I/MC,NBT(175)
      COMMON/M23R/R1(175)
      DO 780 I=1,NN,1
      RHS(I)=FQIE(I)
780       CONTINUE
      IF(MC1)810,810,790
C       INSERT THE BOUNDARY HEAT FLUX.
790       DO 800 I=1,MC1,1
      RHS(NBHF(I))=RHS(NBHF(I))-BHF(NBHF(I))
800       CONTINUE
810       IF(MC2)840,840,820
C       ACCOUNT FOR CONVECTION AT THE BOUNDARY.
820       DO 830 I=1,MC2,1
      BL(I)=SQRT((X(NC1(I))-X(NC2(I)))**2+(Y(NC1(I))-Y(NC2(I)))**2)
      RBL(I)=2.*3.14159*((0.5*(X(NC1(I))+X(NC2(I)))))**2
      IF (NCASE.EQ.2) BL(I)=RBL(I)*BL(I)
      CTINF=H(I)*TA(I)*BL(I)/2.0
      RHS(NC1(I))=RHS(NC1(I))+CTINF
      RHST(NC1(I))=RHST(NC1(I))+CTINF
      RHS(NC2(I))=RHS(NC2(I))+CTINF
      RHST(NC2(I))=RHST(NC2(I))+CTINF
830       CONTINUE
840       CONTINUE
C       INSERT THE BOUNDARY CONDITIONS ON TEMPERATURE.
      DO 900 N=1,MC,1
      I=NBT(N)
      RHS(I)=R1(I)*BT(I)
900       CONTINUE
      RETURN
      END
C       -----------------------------------------------------------------
      SUBROUTINE FORMK
C                          FORMS STIFFNESS MATRIX IN
C                          UPPER TRIANGULAR FORM
      COMMON/M23I/MC,NBT(175)
      COMMON/M23R/R1(175)
      COMMON/M3I/NE,NCN,NDF,NSZF,NBAND
      COMMON/M3R/TM(30,175)
      COMMON/M13I/NODE(300,3)
      COMMON/M13R/TME(3,3)
      DIMENSION ST(5250)
      EQUIVALENCE (ST(1),TM(1,1))
C
C                             ZERO STIFFNESS MATRIX
C
      DO 300 N=1,NSZF
      DO 300 M=1,NBAND
300       TM(M,N)=0
C                             SCAN ELEMENTS
C
      DO 400 N=1,NE
      CALL TSM(N)
C                             RETURN ESTIFM AS STIFFNESS MATRIX
C                             STORE ESTIFM IN SK
C                             FIRST ROWS
```

```
      DO 350 JJ=1,NCN
      NROWB =(NODE(N,JJ)-1)*NDF
      DO 350 J=1,NDF
      IF (NROWB) 350,305,305
305      NROWB =NROWB +1
      I=(JJ-1)*NDF+J
C                             THEN COLUMNS
C
      DO 330 KK=1,NCN
      NCOLB=(NODE(N,KK)-1)*NDF
      DO 320 K=1,NDF
      L=(KK-1)*NDF+K
      NCOL=NCOLB+K+1-NROWB
C                             SKIP STORING IF BELOW BAND
C
C
      IF(NCOL) 320,320,310
310      TM(NCOL,NROWB) = TM(NCOL,NROWB) + TME(I,L)
320      CONTINUE
330      CONTINUE
350      CONTINUE
400      CONTINUE
C                             INSERT BOUNDARY CONDITIONS
C
      DO 500 N=1,MC
      I=NBT(N)
      TM(1,I)=TM(1,I)*1.E15
      R1(I)=TM(1,I)
500      CONTINUE
      DO 1 J=1,NSZF
 169  FORMAT('ROW ',I2)
  166  FORMAT(10F10.2)
 167  FORMAT(//)
1       CONTINUE
C
C                        CHANGE STIFFNESS MATRIX TO CONSECUTIVE
C                                  STORAGE LOCATIONS
C
      M =1
      DO 777 I=1,NSZF
      K =NSZF-I +1
      DO 700 J =1,NBAND
      IF (J-K) 705,705,777
705      ST(M) =TM(J,I)
700      M =M +1
777      CONTINUE
 198   FORMAT(I6,6X,I6)
      RETURN
      END
C
C
C          ------------------------------------------------------------------
      SUBROUTINE MCHB(R,A,M,N,MUD,IOP,EPS,IER,IO)
C         *  IBM SUBROUTINE MCHB(SEE IBM SSP BOOK). USED TO SOLVE MATRIX
C           EQUATION WHEN SQUARE MATRIX IS SYMMETRIC,BANDED,AND POSITIVE
C           DEFINITE. NO NEW INPUT DATA REQUIRED.
      DIMENSION R(300),A(5250)
      DOUBLE PRECISION TOL,SUM,PIV
      IF(IABS(IOP)-3) 1,1,43
1       IF(MUD) 45,2,2
2       MC=MUD+1
      IF(M-MC)46,3,3
3       MR=M-MUD
      IER=0
      IF(IOP)24,4,4
4       IEND=0
      LLDST=MUD
      DO 23 K=1,M
      IST=IEND+1
      IEND=IST+MUD
      J=K-MR
      IF(J)6,6,5
5       IEND=IEND-J
6       IF(J-1)8,8,7
7       LLDST=LLDST-1
8       LMAX=MUD
      J=MC-K
      IF(J)10,10,9
9       LMAX=LMAX-J
10       ID=0
      TOL=A(IST)*EPS
      DO 23 I=IST,IEND
      SUM=0.D0
      IF(LMAX)14,14,11
```

```
11          LL=IST
         LLD=LLDST
         DO 13 L=1,LMAX
         LL=LL-LLD
         LLL=LL+ID
         SUM=SUM+A(LL)*A(LLL)
         B=SUM
         B=ABS(B)
         IF(B.LT.1.0E-35) SUM = 0.0D0
         IF(LLD-MUD)12,13,13
12          LLD=LLD+1
13          CONTINUE
14          SUM=DBLE(A(I))-SUM
         IF(I-IST)15,15,20
15          IF(SUM)47,47,16
16          IF(SUM-TOL)17,17,19
17          IF(IER)18,18,19
18          IER=K-1
19          PIV=DSQRT(SUM)
         A(I)=PIV
         PIV=1.0D0/PIV
         GO TO 21
20          A(I)=SUM*PIV
21          ID=ID+1
         IF(ID-J)23,23,22
22          LMAX=LMAX-1
23          CONTINUE
         IF(IOP)24,44,24
24          ID=N*M
         IEND=IABS(IOP)-2
         IF(IEND)25,35,25
25          IST=1
         LMAX=0
         J=-MR
         LLDST=MUD
         DO 34 K=1,M
         PIV=A(IST)
         IF(PIV)26,48,26
26          PIV=1.0D0/PIV
         DO 30 I=K,ID,M
         SUM=0.0D0
         IF(LMAX)30,30,27
27          LL=IST
         LLL=I
         LLD=LLDST
         DO 29 L=1,LMAX
         LL=LL-LLD
         LLL=LLL-1
         SUM=SUM+A(LL)*R(LLL)
         IF(LLD-MUD)28,29,29
28          LLD=LLD+1
29          CONTINUE
30          R(I)=PIV*(DBLE(R(I))-SUM)
         IF(MC-K)32,32,31
31          LMAX=K
32          IST=IST+MC
         J=J+1
         IF(J)34,34,33
33          IST=IST-J
         LLDST=LLDST-1
34          CONTINUE
         IF(IEND)35,35,44
35          IST=M+(MUD*(M+M-MC))/2+1
         LMAX=0
         K=M
36          IEND=IST-1
         IST=IEND-LMAX
         PIV=A(IST)
         IF(PIV)37,49,37
37          PIV=1.0D0/PIV
         L=IST+1
         DO 40 I=K,ID,M
         SUM=0.0D0
         IF(LMAX)40,40,38
38          LLL=I
         DO 39 LL=L,IEND
         LLL=LLL+1
39          SUM=SUM+A(LL)*R(LLL)
40          R(I)=PIV*(DBLE(R(I))-SUM)
         IF(K-MR) 42,42,41
41          LMAX=LMAX+1
42          K=K-1
         IF(K)44,44,36
43          IER=-1
```

```
      GO TO 44
45       IER=-2
      GO TO 44
46       IER=-3
      GO TO 44
47       IER=-(K+10)
      WRITE (IO,9000) SUM,A(I)
9000        FORMAT(5X,D14.7,E14.7)
      GO TO 44
48       IER = -5
      GO TO 44
49       IER=-6
44       RETURN
      END
C     ------------------------------------------------------------------------------------
      SUBROUTINE ISOTHM(T,NODE,XC,YC,NE,NIT)
      DIMENSION NODE(300,3),T(300),XC(175),YC(175)
      DIMENSION KK(10),TISO(10),X(100),Y(100),XS(100),YS(100)
      DO 350 I=1,NIT,1
      J=0
      READ(13,200,END=1008) TISO(I)
200         FORMAT(F8.2)
      TT=TISO(I)
      DO 57 K=1,NE,1
      N1=NODE(K,1)
      N2=NODE(K,2)
      N3=NODE(K,3)
      T1=T(N1)
      T2=T(N2)
      T3=T(N3)
      X1=XC(N1)
      Y1=YC(N1)
      X2=XC(N2)
      Y2=YC(N2)
      X3=XC(N3)
      Y3=YC(N3)
      TD1=T1-T2
      TD2=T1-T3
      TD3=T2-T3
      IF(TD1)8,60,9
 60   IF(ABS(T1-TT).GT..01) GO TO 28
      J=J+1
      X(J)=X1
      Y(J)=Y1
      J=J+1
      X(J)=X2
      Y(J)=Y2
      GO TO 28
8        TH1=T2
      TC1=T1
      IF(TT-TC1)28,10,10
10       IF(TH1-TT)28,11,11
11       J=J+1
      RATIO=(TH1-TT)/(TH1-TC1)
      X(J)=X2-((X2-X1)*RATIO)
      Y(J)=Y2-((Y2-Y1)*RATIO)
      GO TO 28
9        TH1=T1
      TC1=T2
      IF(TT-TC1)28,12,12
12       IF(TH1-TT)28,13,13
13       J=J+1
      RATIO=(TH1-TT)/(TH1-TC1)
      X(J)=X1-((X1-X2)*RATIO)
      Y(J)=Y1-((Y1-Y2)*RATIO)
28       IF(TD2)14,61,15
 61   IF(ABS(T1-TT).GT..01) GO TO 29
      J=J+1
      X(J)=X1
      Y(J)=Y1
      J=J+1
      X(J)=X3
      Y(J)=Y3
      GO TO 29
14       TH2=T3
      TC2=T1
      IF(TT-TC2)29,16,16
16       IF(TH2-TT)29,17,17
17       J=J+1
      RATIO=(TH2-TT)/(TH2-TC2)
      X(J)=X3-((X3-X1)*RATIO)
      Y(J)=Y3-((Y3-Y1)*RATIO)
      GO TO 29
15       TH2=T1
```

```
      TC2=T3
      IF(TT-TC2)29,18,18
18       IF(TH2-TT)29,19,19
19       J=J+1
      RATIO=(TH2-TT)/(TH2-TC2)
      X(J)=X1-((X1-X3)*RATIO)
      Y(J)=Y1-((Y1-Y3)*RATIO)
29       IF(TD3)20,62,21
 62   IF(ABS(T2-TT).GT..01) GO TO 30
      J=J+1
      X(J)=X2
      Y(J)=Y2
      J=J+1
      X(J)=X3
      Y(J)=Y3
      GO TO 30
20       TH3=T3
      TC3=T2
      IF(TT-TC3)30,22,22
22       IF(TH3-TT)30,23,23
23       J=J+1
      RATIO=(TH3-TT)/(TH3-TC3)
      X(J)=X3-((X3-X2)*RATIO)
      Y(J)=Y3-((Y3-Y2)*RATIO)
      GO TO 30
21       TH3=T2
      TC3=T3
      IF(TT-TC3)30,24,24
24       IF(TH3-TT)30,25,25
25       J=J+1
      RATIO=(TH3-TT)/(TH3-TC3)
      X(J)=X2-((X2-X3)*RATIO)
      Y(J)=Y2-((Y2-Y3)*RATIO)
30       CONTINUE
37       CONTINUE
      K=I
      XS(1)=X(1)
      YS(1)=Y(1)
      DO 80 M1=2,J,1
      KI=0
      DO 90 M2=1,K,1
      IF(ABS(XS(M2)-X(M1)).GT..01) GO TO 90
      IF(ABS(YS(M2)-Y(M1)).GT..01) GO TO 90
      KI=KI+1
90       CONTINUE
      IF(KI.NE.0) GO TO 80
      K=K+1
      XS(K)=X(M1)
      YS(K)=Y(M1)
80       CONTINUE
      KK(I)=K
      WRITE(12,210)TT
210        FORMAT(//,5X,'COORDINATES FOR THE',F8.2,' F ISOTHERM',/,4X,
     X  '--------------------------------------')
      KK(I)=K
      DO 325 N=1,K,1
      WRITE(12,220)XS(N),YS(N)
325        CONTINUE
220        FORMAT(10X,2F10.4)
350        CONTINUE
1009   CONTINUE
      RETURN
      END
```

Table B1. Sample output from FEHEAT — provides initial conditions and final temperature distribution.

```
TOTAL NUMBER OF ELEMENTS =     166

  TOTAL NUMBER OF NODES =   104


     NODE     GLOBAL      GLOBAL
     NO.        X           Y
  -------------------------------
       1      0.0000     -0.1000
       2      0.0383     -0.0924
       3      0.0707     -0.0707
       4      0.0924     -0.0383
       5      0.1000      0.0000
       6      0.0924      0.0383
       7      0.0707      0.0707
       8      0.0383      0.0924
       9      0.0000      0.1000
      10      0.0000     -0.1500
      11      0.0574     -0.1386
      12      0.1061     -0.1061
      13      0.1386     -0.0574
      14      0.1500      0.0000
      15      0.1386      0.0574
      16      0.1061      0.1061
      17      0.0574      0.1386
      18      0.0000      0.1500
      19      0.0000     -0.2000
      20      0.0765     -0.1848
      21      0.1414     -0.1414
      22      0.1848     -0.0765
      23      0.2000      0.0000
      24      0.1848      0.0765
      25      0.1414      0.1414
      26      0.0765      0.1848
      27      0.0000      0.2000
      28      0.0000     -0.3000
      29      0.1148     -0.2772
      30      0.2121     -0.2121
      31      0.2772     -0.1148
      32      0.3000      0.0000
      33      0.2772      0.1148
      34      0.2121      0.2121
      35      0.1148      0.2772
      36      0.0000      0.3000
      37      0.0000     -0.4200
      38      0.1607     -0.3880
      39      0.2970     -0.2970
      40      0.3880     -0.1607
      41      0.4200      0.0000
      42      0.3880      0.1607
      43      0.2970      0.2970
      44      0.1607      0.3880
      45      0.0000      0.4200
      46      0.0000     -0.5800
      47      0.2220     -0.5359
      48      0.4101     -0.4101
      49      0.5359     -0.2220
      50      0.5800      0.0000
      51      0.5359      0.2220
      52      0.4101      0.4101
      53      0.2220      0.5359
      54      0.0000      0.5800
      55      0.0000     -0.7800
      56      0.2985     -0.7206
      57      0.5515     -0.5515
      58      0.7206     -0.2985
      59      0.7800      0.0000
      60      0.7206      0.2985
      61      0.5515      0.5515
      62      0.2985      0.7206
      63      0.0000      0.7800
      64      0.0000     -1.0000
      65      0.3827     -0.9239
      66      0.7071     -0.7071
      67      0.9239     -0.3827
      68      1.0000      0.0000
      69      0.9239      0.3827
      70      0.7071      0.7071
      71      0.3827      0.9239
      72      0.0000      1.0000
      73      0.4142     -1.0000
      74      0.7071     -1.0000
      75      1.0000     -1.0000
      76      1.0000     -0.7071
      77      1.0000     -0.4142
      78      1.0000      0.4142
      79      1.0000      0.7071
      80      1.0000      1.0000
      81      0.7071      1.0000
      82      0.4142      1.0000
      83      0.0000     -1.4000
      84      0.4000     -1.4000
      85      0.8000     -1.4000
      86      1.4000     -1.4000
      87      1.4000     -1.0000
      88      1.4000     -0.6000
      89      1.4000     -0.2000
      90      1.4000      0.2000
      91      1.4000      0.6000
      92      1.4000      1.0000
      93      0.0000     -2.0000
      94      0.4000     -2.0000
      95      0.8000     -2.0000
      96      1.4000     -2.0000
      97      2.0000     -2.0000
      98      2.0000     -1.4000
      99      2.0000     -1.0000
     100      2.0000     -0.6000
     101      2.0000     -0.2000
     102      2.0000      0.2000
     103      2.0000      0.6000
     104      2.0000      1.0000


  ELEMENT  NODE    NODE    NODE    MATL.
    NO.     1       2       3      TYPE
  ---------------------------------------
     1     10       2       1       1
     2     10      11       2       1
     3     11       3       2       1
     4     11      12       3       1
     5     12       4       3       1
     6     12      13       4       1
     7     13       5       4       1
     8     13      14       5       1
     9     14       6       5       1
    10     14      15       6       1
    11     15       7       6       1
    12     15      16       7       1
    13     16       8       7       1
    14     16      17       8       1
    15     17       9       8       1
    16     17      18       9       1
    17     19      11      10       1
    18     19      20      11       1
    19     20      12      11       1
    20     20      21      12       1
    21     21      13      12       1
    22     21      22      13       1
    23     22      14      13       1
    24     22      23      14       1
    25     23      15      14       1
    26     23      24      15       1
    27     24      16      15       1
    28     24      25      16       1
    29     25      17      16       1
    30     25      26      17       1
    31     26      18      17       1
    32     26      27      18       1
    33     28      20      19       1
    34     28      29      20       1
    35     29      21      20       1
    36     29      30      21       1
    37     30      22      21       1
    38     30      31      22       1
```

```
 39     31     23     22     1          123     75     76     66     1
 40     31     32     23     1          124     76     77     66     1
 41     32     24     23     1          125     78     79     70     1
 42     32     33     24     1          126     79     80     70     1
 43     33     25     24     1          127     80     81     70     1
 44     33     34     25     1          128     81     82     70     1
 45     34     26     25     1          129     83     84     64     1
 46     34     35     26     1          130     84     73     64     1
 47     35     27     26     1          131     84     74     73     1
 48     35     36     27     1          132     84     85     74     1
 49     37     29     28     1          133     85     75     74     1
 50     37     38     29     1          134     85     86     75     1
 51     38     30     29     1          135     86     87     75     1
 52     38     39     30     1          136     75     87     76     1
 53     39     31     30     1          137     87     88     76     1
 54     39     40     31     1          138     76     88     77     1
 55     40     32     31     1          139     88     89     77     1
 56     40     41     32     1          140     77     89     68     1
 57     41     33     32     1          141     89     90     68     1
 58     41     42     33     1          142     68     90     78     1
 59     42     34     33     1          143     90     91     78     1
 60     42     43     34     1          144     78     91     79     1
 61     43     35     34     1          145     91     92     79     1
 62     43     44     35     1          146     79     92     80     1
 63     44     36     35     1          147     93     94     83     1
 64     44     45     36     1          148     94     84     83     1
 65     46     38     37     1          149     94     95     84     1
 66     46     47     38     1          150     95     85     84     1
 67     47     39     38     1          151     95     96     85     1
 68     47     48     39     1          152     96     86     85     1
 69     48     40     39     1          153     96     97     86     1
 70     48     49     40     1          154     97     98     86     1
 71     49     41     40     1          155     86     98     87     1
 72     49     50     41     1          156     98     99     87     1
 73     50     42     41     1          157     87     99     88     1
 74     50     51     42     1          158     99    100     88     1
 75     51     43     42     1          159     88    100     89     1
 76     51     52     43     1          160    100    101     89     1
 77     52     44     43     1          161     89    101    102     1
 78     52     53     44     1          162     89    102     90     1
 79     53     45     44     1          163     90    102    103     1
 80     53     54     45     1          164     90    103     91     1
 81     55     47     46     1          165     91    103    104     1
 82     55     56     47     1          166     91    104     92     1
 83     56     48     47     1
 84     56     57     48     1            B.C.   NODE   TEMP
 85     57     49     48     1            NO.    NO.    (F)
 86     57     58     49     1          -----------------------------
 87     58     50     49     1              1      1   100.0000
 88     58     59     50     1              2      2   100.0000
 89     59     51     50     1              3      3   100.0000
 90     59     60     51     1              4      4   100.0000
 91     60     52     51     1              5      5   100.0000
 92     60     61     52     1              6      6   100.0000
 93     61     53     52     1              7      7   100.0000
 94     61     62     53     1              8      8   100.0000
 95     62     54     53     1              9      9   100.0000
 96     62     63     54     1             10     80    10.0000
 97     64     56     55     1             11     81    10.0000
 98     64     65     56     1             12     82    10.0000
 99     65     57     56     1             13     72    10.0000
100     65     66     57     1             14     93    10.0000
101     66     58     57     1             15     94    10.0000
102     66     67     58     1             16     95    10.0000
103     67     59     58     1             17     96    10.0000
104     67     68     59     1             18     97    10.0000
105     68     60     59     1             19     98    10.0000
106     68     69     60     1             20     99    10.0000
107     69     61     60     1             21    100    10.0000
108     69     70     61     1             22    101    10.0000
109     70     62     61     1             23    102    10.0000
110     70     71     62     1             24    103    10.0000
111     71     63     62     1             25    104    10.0000
112     71     72     63     1             26     92    10.0000
113     64     73     65     1              THE BANDWIDTH IS      [illegible]3
114     73     66     65     1
115     65     77     67     1           NODE   TEMPERATURE (F)
116     77     69     67     1          ----------------------------
117     66     78     69     1             1          100.00
118     76     79     69     1             2          100.00
119     79     82     71     1             3          100.00
120     82     72     71     1             4          100.00
121     73     74     66     1
122     74     75     66     1
```

```
   5          100.00
   6          100.00
   7          100.00
   8          100.00
   9          100.00
  10           87.52
  11           87.48
  12           87.36
  13           87.17
  14           86.92
  15           86.61
  16           86.31
  17           86.10
  18           86.02
  19           78.66
  20           78.59
  21           78.40
  22           78.05
  23           77.59
  24           77.05
  25           76.51
  26           76.08
  27           75.92
  28           66.38
  29           66.27
  30           65.97
  31           65.41
  32           64.63
  33           63.65
  34           62.64
  35           61.80
  36           61.47
  37           56.31
  38           56.19
  39           55.79
  40           55.06
  41           53.97
  42           52.54
  43           50.91
  44           49.50
  45           48.91
  46           48.81
  47           46.69
  48           46.26
  49           45.37
  50           43.97
  51           42.00
  52           39.53
  53           37.10
  54           36.02
  55           38.21
  56           34.16
  57           37.77
  58           35.78
  59           35.12
  60           32.63
  61           29.02
  62           24.92
  63           22.86
  64           31.01
  65           31.15
  66           30.89
  67           29.86
  68           28.03
  69           25.25
  70           20.40
  71           13.79
  72           10.00
  73           29.93
  74           25.52
  75           21.63
  76           25.00
  77           27.71
  78           23.05
  79           15.96
  80           10.00
  81           10.00
  82           10.00
  83           21.24
  84           20.33
  85           18.13
  86           14.15
  87           16.72
  88           18.72
  89           19.37
  90           13.02
  91           14.66
  92           10.00
  93           10.00
  94           10.00
  95           10.00
  96           10.00
  97           10.00
  98           10.00
  99           10.00
 100           10.00
 101           10.00
 102           10.00
 103           10.00
 104           10.00

 COORDINATES FOR THL    10.00 F ISOTHERM
-----------------------------------------
           0.0000     1.0000
           0.4142     1.0000
           1.0000     1.0000
           0.7071     1.0000
           1.4000     1.0000
           0.0000    -2.0000
           0.4000    -2.0000
           0.6000    -2.0000
           1.4000    -2.0000
           2.0000    -2.0000
           2.0000    -1.7000
           2.0000    -1.0000
           2.0000    -0.6000
           2.0000    -0.2000
           2.0000     0.2000
           2.0000     0.6000
           2.0000     1.0000

 COORDINATES FOR THE    30.00 F ISOTHERM
-----------------------------------------
           0.5373     0.4830
           0.5333     0.5363
           0.5116     0.5476
           0.2666     0.6436
           0.1618     0.6562
           0.0000     0.6714
           0.8944    -0.4268
           0.9198    -0.3810
           0.9389     0.0000
           0.8806     0.1276
           0.7931     0.3285
           0.2018    -1.0000
           0.3990    -0.9633
           0.5736    -0.8406
           0.7833    -0.6249
           0.7071    -0.7558
           0.7354    -0.7354
           0.7516    -0.7071
           0.0000    -1.0415
           0.0380    -1.0380

 COORDINATES FOR THE    50.00 F ISOTHERM
-----------------------------------------
           0.2088     0.3559
           0.1588     0.3835
           0.0000     0.4096
           0.0547    -0.5146
           0.0000    -0.5262
           0.2006    -0.4844
           0.2493    -0.4470
           0.3657    -0.3657
           0.4007    -0.3041
           0.4652    -0.1927
           0.4735    -0.1025
           0.4835     0.0000
           0.4450     0.1130
           0.4237     0.1755
           0.3215     0.2893
           0.3061     0.3061
 COORDINATES FOR THE    70.00 F ISOTHERM
-----------------------------------------
              0.0227    -0.2658
              0.0000    -0.2705
              0.1032    -0.2493
              0.1230    -0.2355
              0.1892    -0.1892
              0.2030    -0.1668
              0.2436    -0.1009
              0.2481    -0.0715
              0.2586     0.0000
              0.2502     0.0331
              0.2334     0.0967
              0.2101     0.1279
              0.1746     0.1746
              0.1378     0.1971
              0.0928     0.2241
              0.0481     0.2324
              0.0000     0.2410

 COORDINATES FOR THE    90.00 F ISOTHERM
-----------------------------------------
              0.0076    -0.1386
              0.0536    -0.1293
              0.0987    -0.0987
              0.1284    -0.0532
              0.1382     0.0000
              0.1269     0.0526
              0.0966     0.0966
              0.0520     0.1256
              0.0413     0.1278
              0.0000     0.1358

 COORDINATES FOR THE   100.00 F ISOTHERM
-----------------------------------------
              0.0383    -0.0924
              0.0707    -0.0707
              0.0924    -0.0383
              0.0924     0.0383
              0.0707     0.0707
              0.0383     0.0924
              0.0000     0.1000
```

Table B2. Sample input to FEHEAT—all files must be typed in by user.

a. File ED.

1	10	2	1	1
2	10	11	2	1
3	11	3	2	1
4	11	12	3	1
5	12	4	3	1
6	12	13	4	1
7	13	5	4	1
8	13	14	5	1
9	14	6	5	1
10	14	15	6	1
11	15	7	6	1
12	15	16	7	1
13	16	8	7	1
14	16	17	8	1
15	17	9	8	1
16	17	18	9	1
17	19	11	10	1
18	19	20	11	1
19	20	12	11	1
20	20	21	12	1
21	21	13	12	1
22	21	22	13	1
23	22	14	13	1
24	22	23	14	1
25	23	15	14	1
26	23	24	15	1
27	24	16	15	1
28	24	25	16	1
29	25	17	16	1
30	25	26	17	1
31	26	18	17	1
32	26	27	18	1
33	28	20	19	1
34	28	29	20	1
35	29	21	20	1
36	29	30	21	1
37	30	22	21	1
38	30	31	22	1
39	31	23	22	1
40	31	32	23	1
41	32	24	23	1
42	32	33	24	1
43	33	25	24	1
44	33	34	25	1
45	34	26	25	1
46	34	35	26	1
47	35	27	26	1
48	35	36	27	1
49	37	29	28	1
50	37	38	29	1
51	38	30	29	1
52	38	39	30	1
53	39	31	30	1
54	39	40	31	1
55	40	32	31	1
56	40	41	32	1
57	41	33	32	1
58	41	42	33	1
59	42	34	33	1
60	42	43	34	1
61	43	35	34	1
62	43	44	35	1
63	44	36	35	1
64	44	45	36	1
65	46	38	37	1
66	46	47	38	1
67	47	39	38	1
68	47	48	39	1
69	48	40	39	1
70	48	49	40	1
71	49	41	40	1
72	49	50	41	1
73	50	42	41	1
74	50	51	42	1
75	51	43	42	1
76	51	52	43	1
77	52	44	43	1
78	52	53	44	1
79	53	45	44	1
80	53	54	45	1
81	55	47	46	1
82	55	56	47	1
83	56	48	47	1
84	56	57	48	1
85	57	49	48	1
86	57	58	49	1
87	58	50	49	1
88	58	59	50	1
89	59	51	50	1
90	59	60	51	1
91	60	52	51	1
92	60	61	52	1
93	61	53	52	1
94	61	62	53	1
95	62	54	53	1
96	62	63	54	1
97	64	56	55	1
98	64	65	56	1
99	65	57	56	1
100	65	66	57	1
101	66	58	57	1
102	66	67	58	1
103	67	59	58	1
104	67	68	59	1
105	68	60	59	1
106	68	69	60	1
107	69	61	60	1
108	69	70	61	1
109	70	62	61	1
110	70	71	62	1
111	71	63	62	1
112	71	72	63	1
113	64	73	65	1
114	73	66	65	1
115	66	77	67	1
116	77	68	67	1
117	68	78	69	1
118	78	70	69	1
119	70	82	71	1
120	82	72	71	1
121	73	74	66	1
122	74	75	66	1
123	75	76	66	1
124	76	77	66	1
125	78	79	70	1
126	79	80	70	1
127	80	81	70	1
128	81	82	70	1
129	83	84	64	1
130	84	73	64	1
131	94	74	73	1
132	84	85	74	1
133	85	75	74	1
134	85	86	75	1
135	86	87	75	1
136	75	87	76	1
137	87	88	76	1
138	76	88	77	1
139	88	89	77	1
140	77	89	68	1
141	80	90	68	1
142	68	90	78	1
143	90	91	78	1
144	78	91	79	1
145	91	92	79	1
146	79	92	80	1
147	93	94	83	1
148	94	84	83	1
149	94	95	84	1
150	95	85	84	1
151	95	96	85	1
152	96	86	85	1
153	96	97	86	1
154	97	98	86	1
155	86	98	87	1
156	98	99	87	1
157	87	99	88	1
158	99	100	88	1
159	88	100	89	1
160	100	101	89	1
161	89	101	102	1
162	89	102	90	1
163	90	102	103	1
164	90	103	91	1
165	91	103	104	1
166	91	104	92	1

Table B2 (cont'd).

b. File NPD.

1	0.0000	-0.1000
2	0.0383	-0.0924
3	0.0707	-0.0707
4	0.0924	-0.0383
5	0.1000	0.0000
6	0.0924	0.0383
7	0.0707	0.0707
8	0.0383	0.0924
9	0.0000	0.1000
10	0.0000	-0.1500
11	0.0574	-0.1386
12	0.1061	-0.1061
13	0.1386	-0.0574
14	0.1500	0.0000
15	0.1386	0.0574
16	0.1061	0.1061
17	0.0574	0.1386
18	0.0000	0.1500
19	0.0000	-0.2000
20	0.0765	-0.1848
21	0.1414	-0.1414
22	0.1848	-0.0765
23	0.2000	0.0000
24	0.1848	0.0765
25	0.1414	0.1414
26	0.0765	0.1848
27	0.0000	0.2000
28	0.0000	-0.3000
29	0.1148	-0.2772
30	0.2121	-0.2121
31	0.2772	-0.1149
32	0.3000	0.0000
33	0.2772	0.1148
34	0.2121	0.2121
35	0.1148	0.2772
36	0.0000	0.3000
37	0.0000	-0.4200
38	0.1607	-0.3880
39	0.2970	-0.2970
40	0.3880	-0.1607
41	0.4200	0.0000
42	0.3880	0.1607
43	0.2970	0.2970
44	0.1607	0.3880
45	0.0000	0.4200
46	0.0000	-0.5800
47	0.2220	-0.5359
48	0.4101	-0.4101
49	0.5359	-0.2220
50	0.5800	0.0000
51	0.5359	0.2220
52	0.4101	0.4101
53	0.2220	0.5359
54	0.0000	0.5800
55	0.0000	-0.7800
56	0.2985	-0.7206
57	0.5515	-0.5515
58	0.7206	-0.2985
59	0.7800	0.0000
60	0.7206	0.2985
61	0.5515	0.5515
62	0.2985	0.7206
63	0.0000	0.7800
64	0.0000	-1.0000
65	0.3827	-0.9239
66	0.7071	-0.7071
67	0.9239	-0.3827
68	1.0000	0.0000
69	0.9239	0.3827
70	0.7071	0.7071
71	0.3827	0.9239
72	0.0000	1.0000
73	0.4142	-1.0000
74	0.7071	-1.0000
75	1.0000	-1.0000
76	1.0000	-0.7071
77	1.0000	-0.4142
78	1.0000	0.4142
79	1.0000	0.7071
80	1.0000	1.0000
81	0.7071	1.0000
82	0.4142	1.0000
83	0.0000	-1.4000
84	0.4000	-1.4000
85	0.8000	-1.4000
86	1.4000	-1.4000
87	1.4000	-1.0000
88	1.4000	-0.6000
89	1.4000	-0.2000
90	1.4000	0.2000
91	1.4000	0.6000
92	1.4000	1.0000
93	0.0000	-2.0000
94	0.4000	-2.0000
95	0.8000	-2.0000
96	1.4000	-2.0000
97	2.0000	-2.0000
98	2.0000	-1.4000
99	2.0000	-1.0000
100	2.0000	-0.6000
101	2.0000	-0.2000
102	2.0000	0.2000
103	2.0000	0.6000
104	2.0000	1.0000

c. File BCT.

```
 1      1   100.0000
 2      2   100.0000
 3      3   100.0000
 4      4   100.0000
 5      5   100.0000
 6      6   100.0000
 7      7   100.0000
 8      8   100.0000
 9      9   100.0000
10     80    10.0000
11     81    10.0000
12     82    10.0000
13     72    10.0000
14     93    10.0000
15     94    10.0000
16     95    10.0000
17     96    10.0000
18     97    10.0000
19     98    10.0000
20     99    10.0000
21    100    10.0000
22    101    10.0000
23    102    10.0000
24    103    10.0000
25    104    10.0000
26     92    10.0000
```

d. File QUAN.

```
104    166    26     0     0     0     6
```

e. File TIOT.

```
 10.00
 30.00
 50.00
 70.00
 90.00
100.00
```

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Albert, M.R.
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